Pseudorandom Functions and Lattices

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Symbolic Computations and Post-Quantum Crypto
8 December 2011

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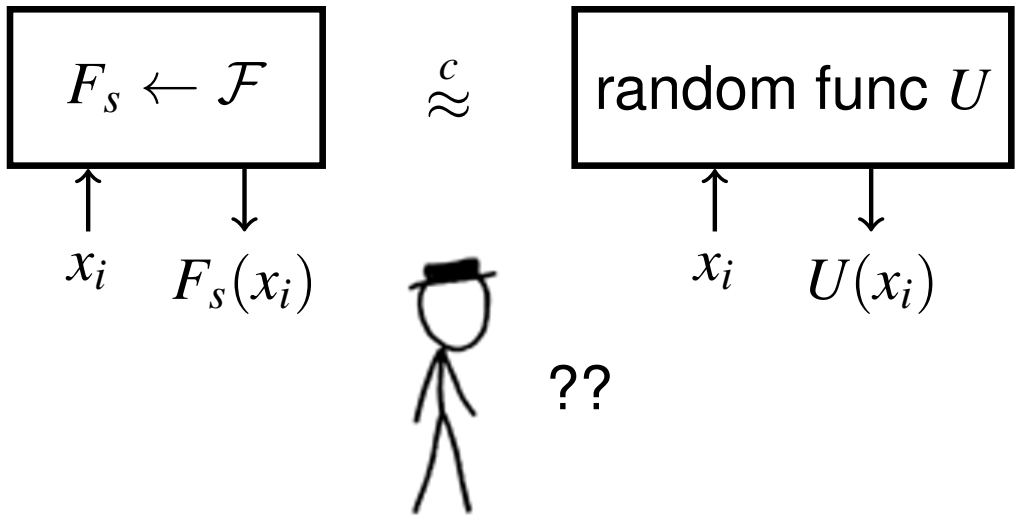

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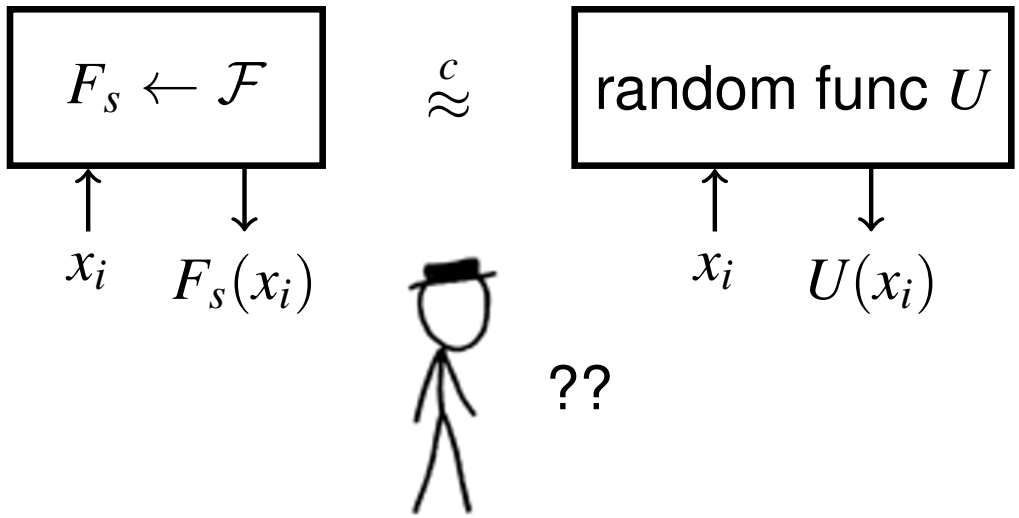

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- Countless applications in symmetric cryptography: (efficient) encryption, identification, authentication, ...

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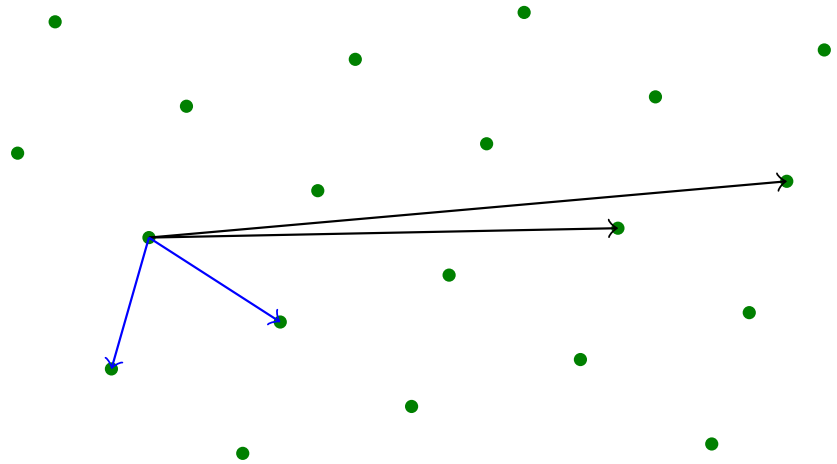
   - ✗ No "post-quantum" construction under standard assumptions

Why Not Try Lattices?



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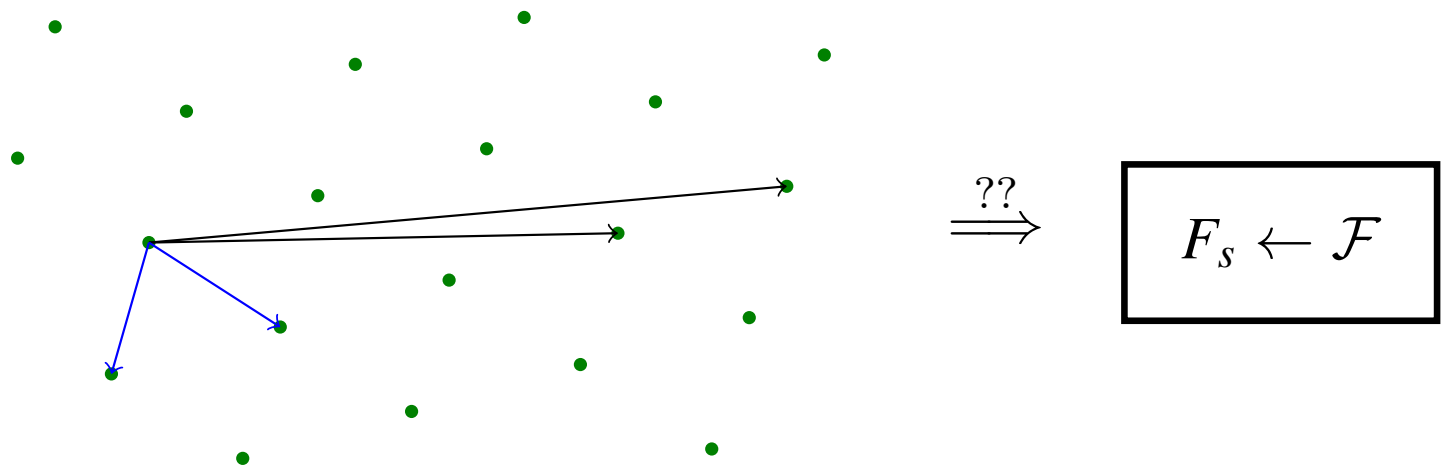

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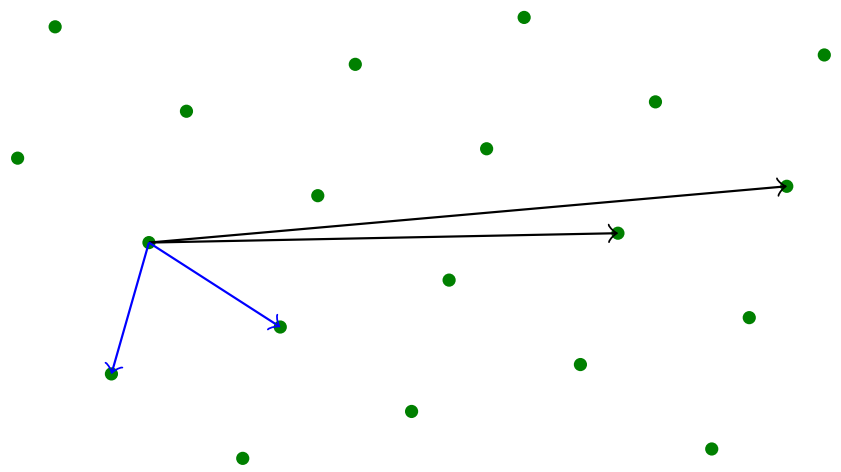

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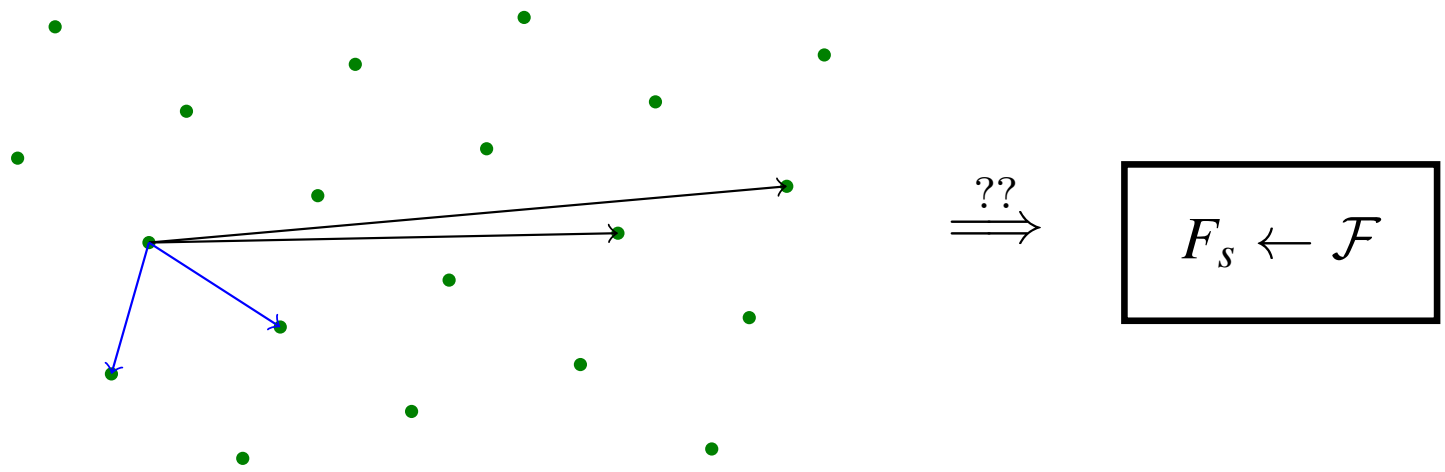

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✗✗ We don't even have practical PRGs from lattices: biased errors

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   Also gives more practical PRGs, GGM-type PRFs, encryption, ...

Synthesizers and PRFs [NaorReingold'95]

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- A deterministic function $S\colon D \times D \to D$ s.t. for any $m = \text{poly}$:
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- Alternative view: an (almost) length-squaring PRG with locality:
  maps $D^{2m} \to D^{m^2}$, and each output depends on only 2 inputs.

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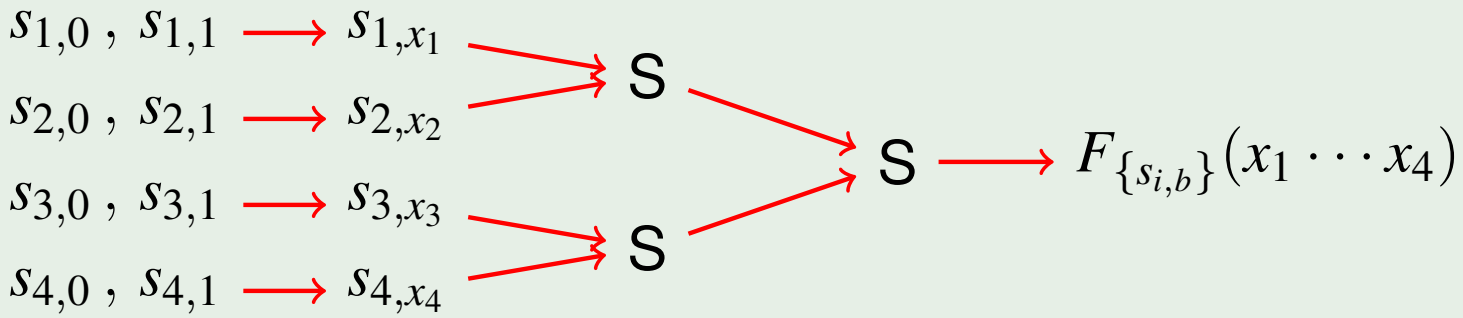

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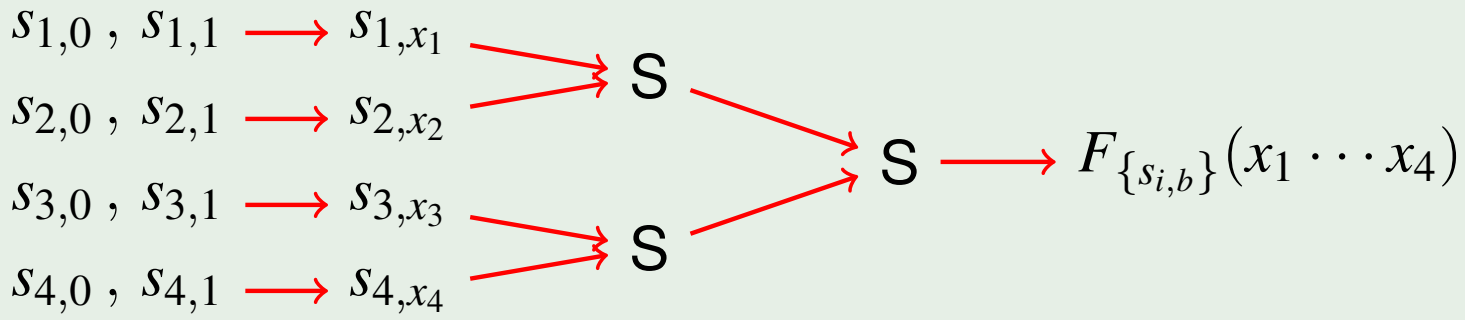

- Security: the queries $F_\ell(x_\ell)$ and $F_r(x_r)$ define (pseudo)random inputs $a_1, a_2, \ldots \in D$ and $b_1, b_2, \ldots \in D$ for synthesizer S.

(Ring) Learning With Errors (RLWE) [Regev'05,LPR'10]

- For (e.g.) n a power of 2, define "cyclotomic" polynomial rings

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✗ Where do $e_{i,j}$ come from?

Synthesizer must be deterministic...

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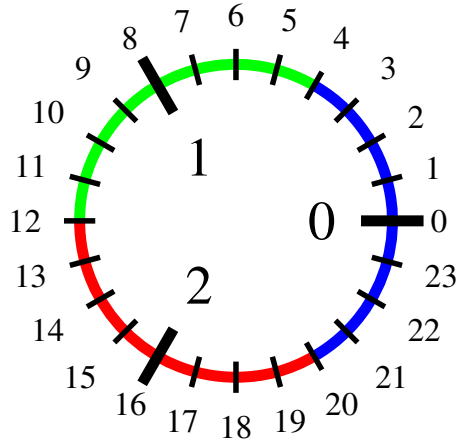

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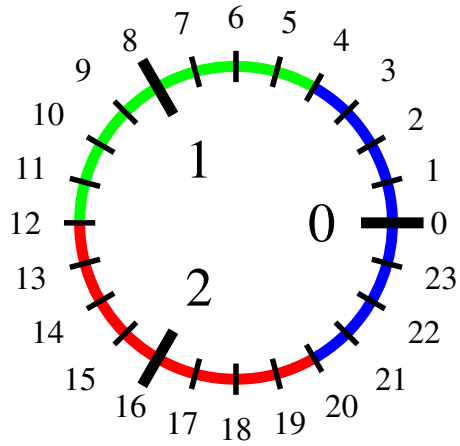

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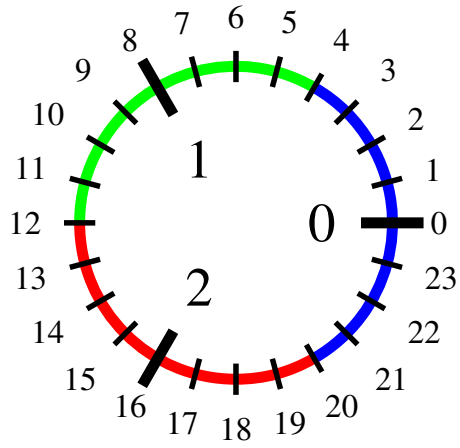

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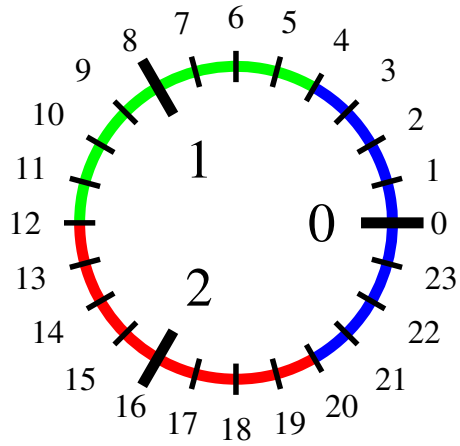

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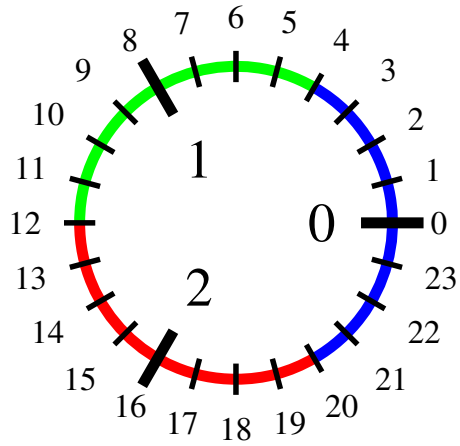

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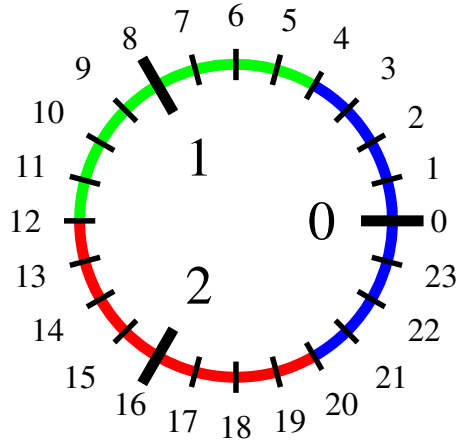

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  Main idea: w.h.p. $\quad (a, \lfloor a \cdot s + e \rceil_p) = (a, \lfloor a \cdot s \rceil_p)$

  and $\quad (a, \lfloor \text{Unif}(\mathbb{Z}_q) \rceil_p) = (a, \text{Unif}(\mathbb{Z}_p))$

LWR-Based Synthesizer & PRF

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- Depth $d = \lg k$ tree of LWR synthesizers:

$$F_{\{s_{i,b}\}}(x_1 \cdots x_8) = \left\lfloor \left\lfloor \left\lfloor s_{1,x_1} \cdot s_{2,x_2} \right\rceil_{q_2} \cdot \left\lfloor s_{3,x_3} \cdot s_{4,x_4} \right\rceil_{q_2} \right\rceil_{q_1} \cdot \left\lfloor \left\lfloor s_{5,x_5} \cdot s_{6,x_6} \right\rceil_{q_2} \cdot \left\lfloor s_{7,x_7} \cdot s_{8,x_8} \right\rceil_{q_2} \right\rceil_{q_1} \right\rceil_{q_0}$$

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Has small(ish) TC^0 circuit, via CRT and reduction to subset-sum.

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- Repeat for $s_2, s_3, \ldots$ until $F''''''(x) = \lfloor a_x \rceil_p =$ Uniform func. $\square$

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Thanks!

Full paper: ePrint report #2011/401