Algorithmic problems on compressed words

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  Examples in: combinatorial group theory, computational topology, program analysis, verification, ...

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Definition (Straight-line program (SLP))

An SLP over the alphabet Γ is a sequence of definitions

$$\mathbb{A} = \langle A_i := \alpha_i \rangle_{0 \le i \le n},$$

where either $\alpha_i \in \Gamma$ or $\alpha_i = A_j A_k$ for some $j, k > i$.

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Alternatively: a context-free grammar that generates exactly one string.

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Algorithms on SLP-compressed strings

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Plandowski's algorithm uses combinatorics on words, in particular the Periodicity Lemma of Fine and Wilf.

Improvements of Plandowski's result

Gasieniec, Karpinski, Miyazaki, Plandowski, Rytter, Shinohara, Takeda (mid 90's)

The following problem can be solved in polynomial time (fully compressed pattern matching):

INPUT: SLPs $\mathbb{P}, \mathbb{T}$

QUESTION: Is $\mathsf{val}(\mathbb{P})$ a factor of $\mathsf{val}(\mathbb{T})$, i.e., $\exists u, v : \mathsf{val}(\mathbb{T}) = u \, \mathsf{val}(\mathbb{P}) \, v$?

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The best known algorithm has a running time of $\mathcal{O}(|\mathbb{P}| \cdot |\mathbb{T}|^2)$ (Lifshits 2006).

Parsing compressed strings: finite automata

Plandowski, Rytter 1999

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INPUT: A nondeterministic automaton A and an SLP $\mathbb{B}$.

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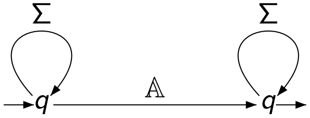

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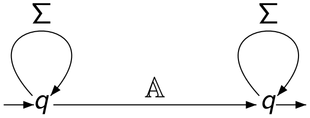

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The size $|\mathcal{A}|$ of the compressed automaton $\mathcal{A}$ is

$$|\mathcal{A}| = \sum_{p \xrightarrow{\mathbb{A}} q} |\mathbb{A}|.$$

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Compressed membership for compressed automata:

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QUESTION: $\mathsf{val}(\mathbb{B}) \in L(\mathcal{A})$?

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- Compressed membership for compressed automata is in PSPACE.
- Compressed membership for compressed automata over a unary alphabet is NP-complete.

Parsing compressed strings: compressed finite automata

Conjecture (Plandowski, Rytter 1999)

- Compressed membership for compressed automata is NP-complete (for every alphabet size).
- Compressed membership for compressed deterministic automata belongs to P.

  A compressed automaton $\mathcal{A}$ is deterministic, if for all transitions $p \xrightarrow{\mathbb{A}} q$, $p \xrightarrow{\mathbb{B}} r$ that start in the same state p, neither $\mathrm{val}(\mathbb{A})$ is a prefix of $\mathrm{val}(\mathbb{B})$ nor $\mathrm{val}(\mathbb{B})$ is a prefix of $\mathrm{val}(\mathbb{A})$.

Parsing compressed strings: pushdown automata

Theorem (L 2010)

The following problem is PSPACE-complete:

INPUT: A pushdown automaton A and an SLP $\mathbb{B}$.

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The proof uses a characterization of PSPACE based on leaf languages (Hertrampf, Lautemann, Schwentick, Vollmer, Wagner; 1993).

Other hard problems for compressed strings

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PP is the class of all problems A for which there exists a probabilitstic polynomial time machine M such that

$$\forall x : x \in A \iff \text{Prob}[M \text{ accepts } x] > 1/2$$

Toda 1991: P^{PP} contains the polynomial time hierarchy.

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Assume that the compressed word problem for G can be solved in polynomial time.

Then, for every finitely generated subgroup of $\mathrm{Aut}(G)$ the (standard) word problem can be solved in polynomial time.

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Closure properties of the class of groups with polynomial time CWP:

- going to f.g. subgroups
- finite extensions
- graph products (in particular free and direct products)

Application in computational group theory

Classes of groups, where CWP can be solved in polynomial time:

- graph groups (in particular free groups)
- nilpotent groups
- fully residually-free groups (Macdonald 2010)
- Coxeter groups

Closure properties of the class of groups with polynomial time CWP:

- going to f.g. subgroups
- finite extensions
- graph products (in particular free and direct products)
- HNN-extensions and amalgamated free products over finite groups

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- Compressed word problem for braid groups, polycyclic groups, and finitely generated metabelian groups