New Algorithms for Learning in Presence of Errors

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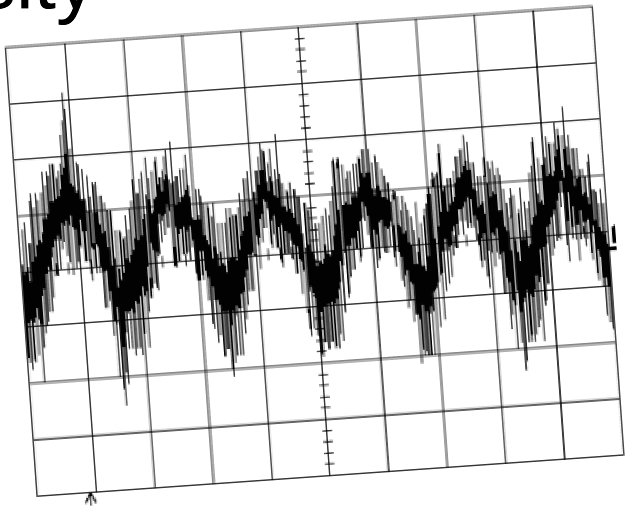
Princeton University



Hard(?) Problems

Treasure shall be found at u

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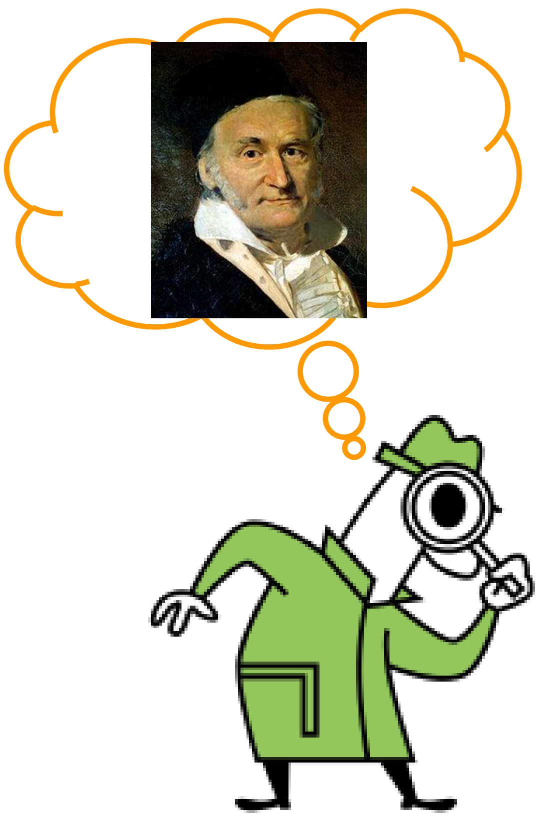

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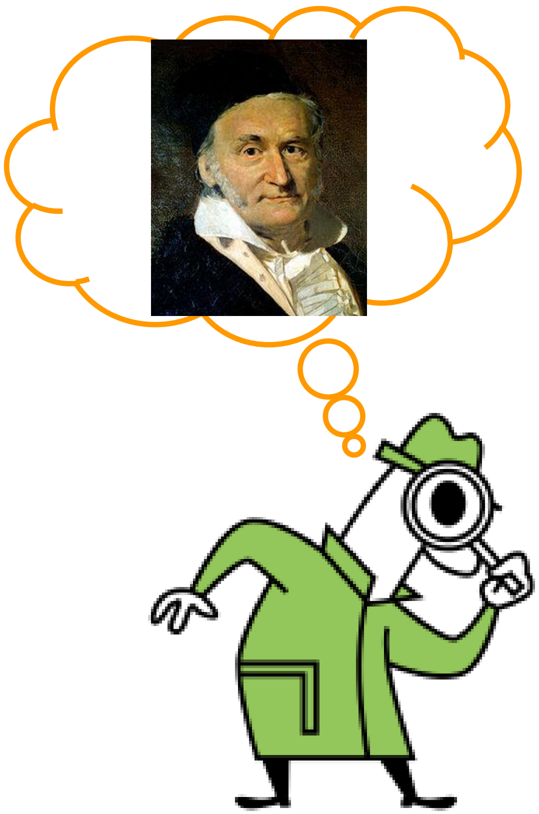

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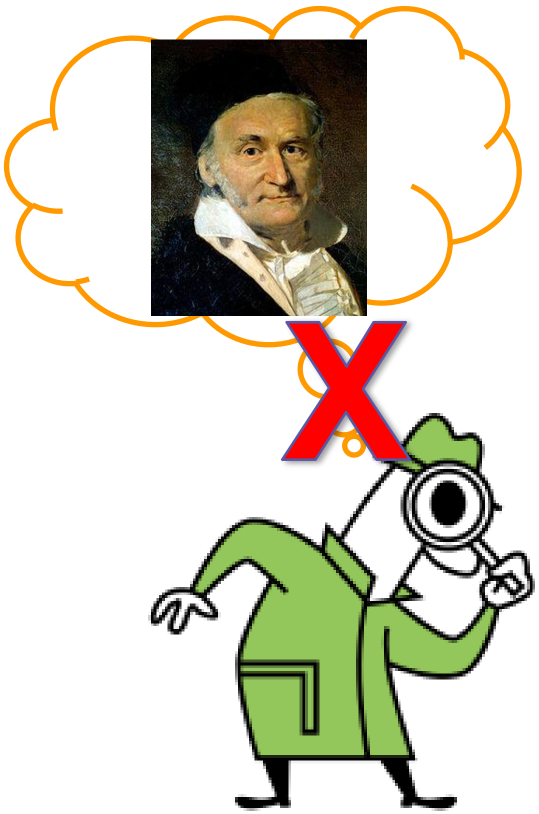

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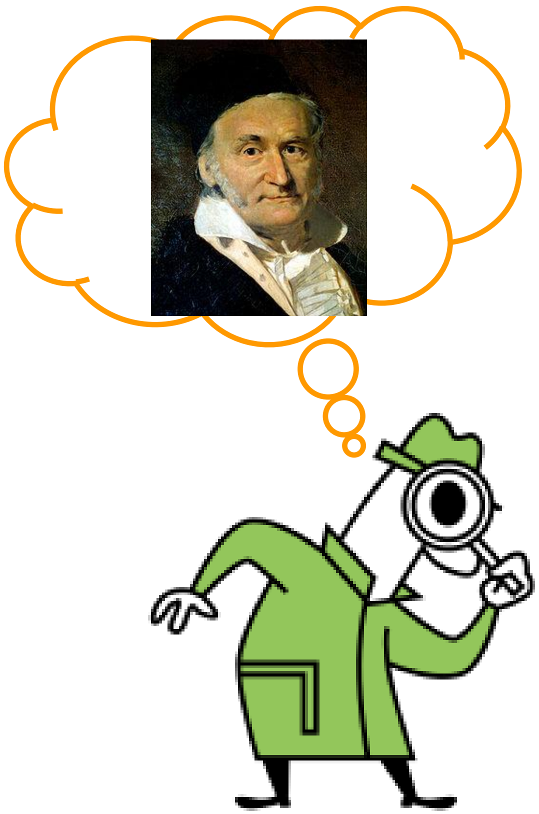

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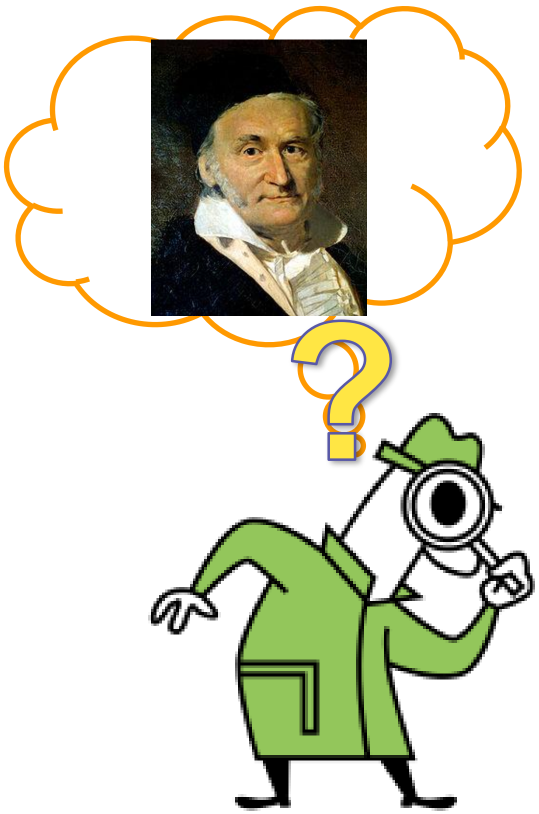

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- Used in designing public-key crypto [Alekhnovich'03]

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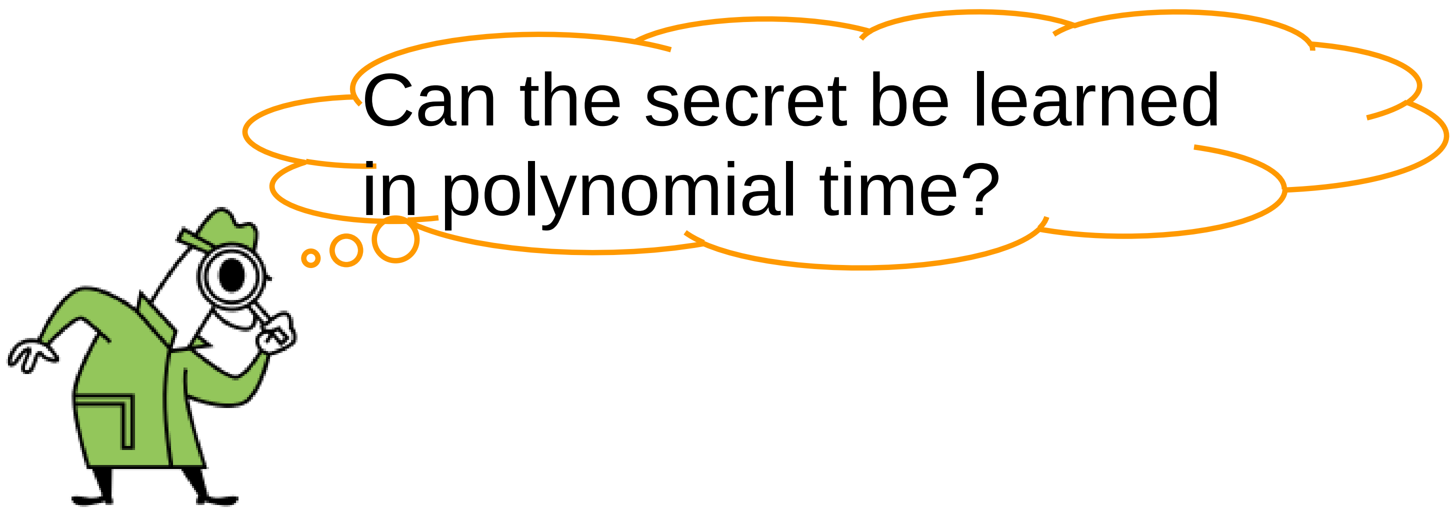

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  - Subexp algorithm when noise $< n^{1/2}$
  - Open problem since [Regev'05]
- Majority of 3 parities
  - Can inverse with $O(n^2 \log n)$ queries.
  - Pseudorandom generator purposed in [ABW'10]

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- "No 3 consecutive wrong inner-products"
  - $P(c) = c_1c_2c_3+c_2c_3c_4+\ldots+c_{m-2}c_{m-1}c_m = 0$

Notations

- Subscripts are used for indexing vectors
  - u_i, c_i
- Superscripts are used for a list of vectors
  - a^i
- High dimensional vectors are indexed like $Z_{i,j,k}$
- a, b are known constants, u, c are unknown constants used in analysis, x, y, Z are variables in equations.

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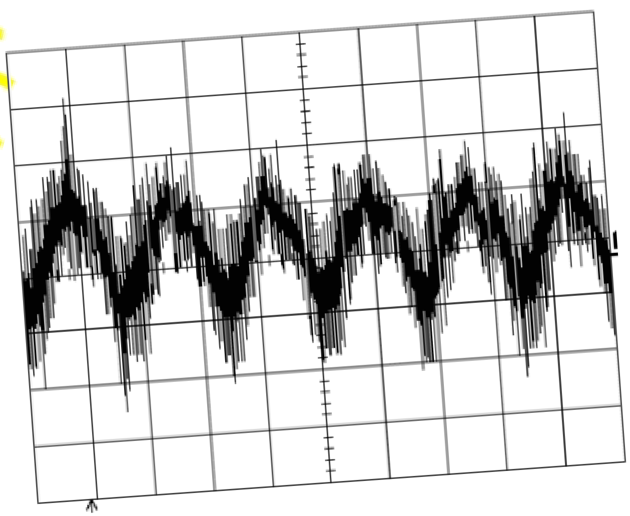

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Linear Equations over y

(**) = L((*))

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- Coming up: This is the only solution to the system of linear equations

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$Z^3 = 0 \Leftrightarrow$ y is the canonical solution

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- Schwartz-Zippel
  - The polynomial is non-zero w.p. at least 2^{-d}

Main Lemma → Theorem

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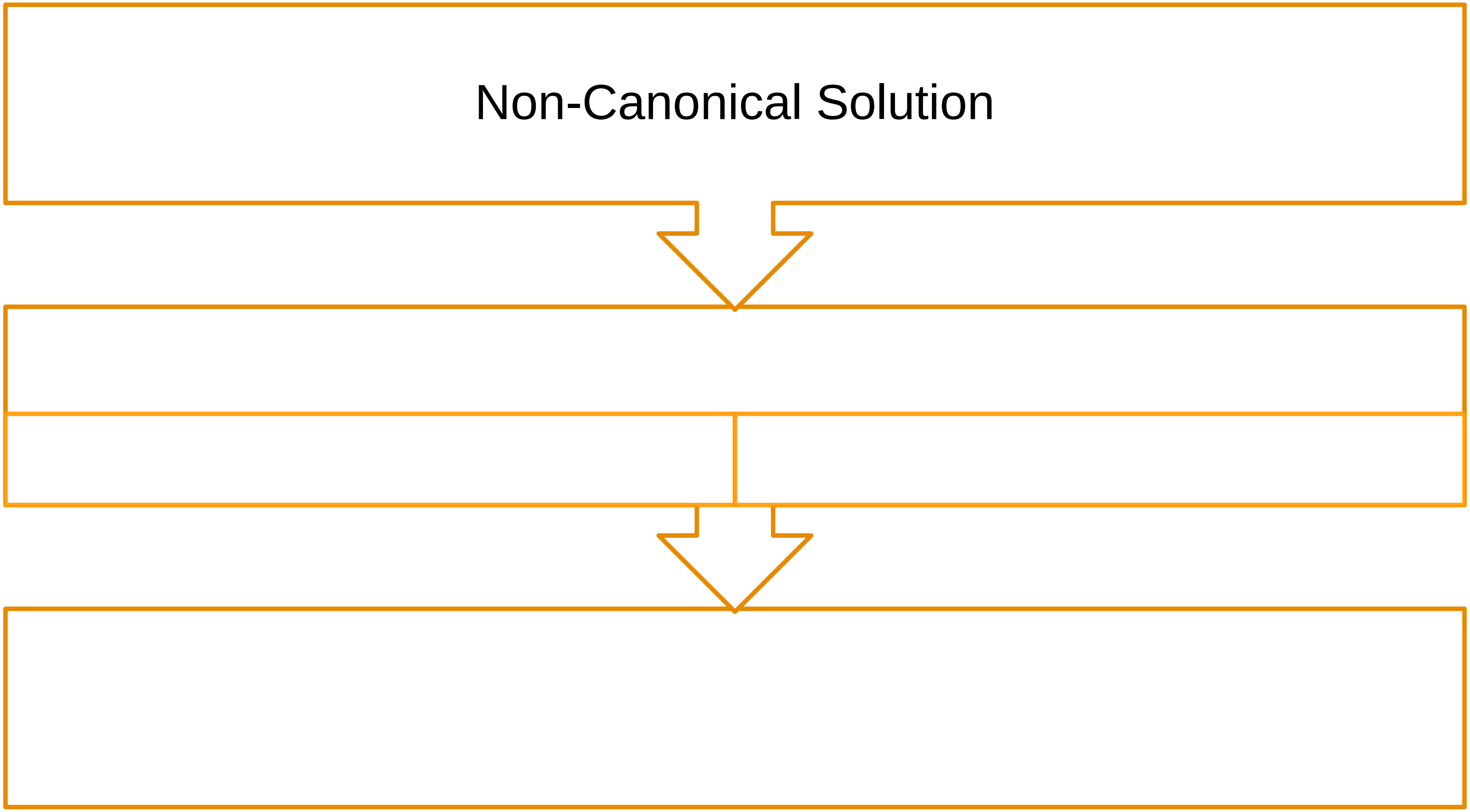

Main Lemma → Theorem

Non-Canonical Solution

↓

Non-zero Z^3 vector, Poly(a) = 0 for all equations

↓

Main Lemma → Theorem

Non-Canonical Solution

Non-zero Z^3 vector, Poly(a) = 0 for all equations

Schwartz-Zippel

Union Bound

Main Lemma → Theorem

Non-Canonical Solution

↓

Non-zero Z^3 vector, Poly(a) = 0 for all equations

Schwartz-Zippel | Union Bound

↓

Low Probability

Main Lemma → Theorem

Non-Canonical Solution

↓

Non-zero Z^3 vector, Poly(a) = 0 for all equations

Schwartz-Zippel | Union Bound

↓

With High Probability

Main Lemma → Theorem

No Non-Canonical Solutions

Non-zero Z^3 vector, F[illegible]) = 0 for all equations

Schwartz-Zippel	Union Bound

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□

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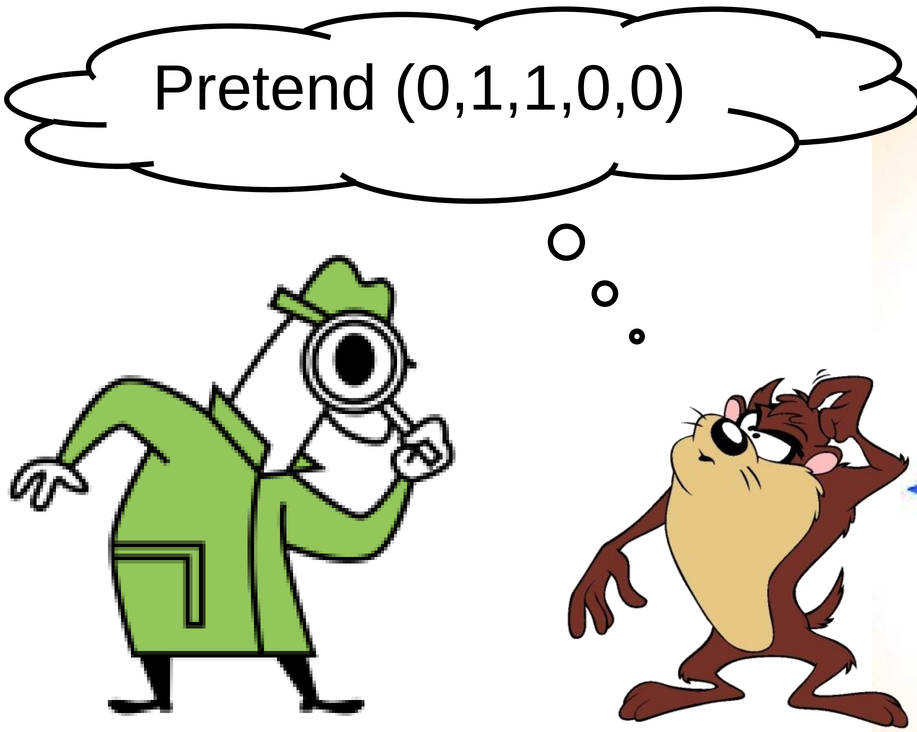

Adversarial Noise

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  the secret can be learned in $n^{O(m)}$ time
  otherwise no algorithm can learn the secret

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Canonical Solution: still satisfied
Non-Canonical: cannot be satisfied because noise c = (0,0,0) is always present

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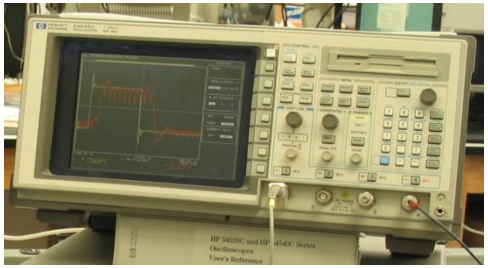

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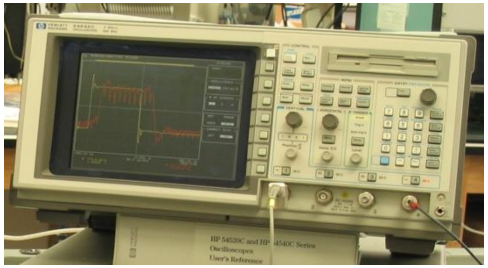

- Provable reduction from worst case lattice problems

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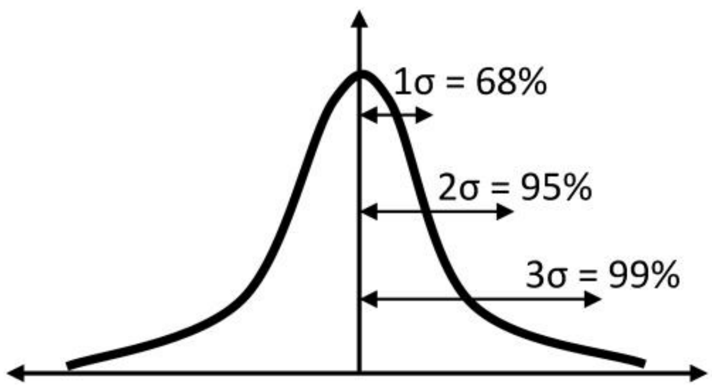

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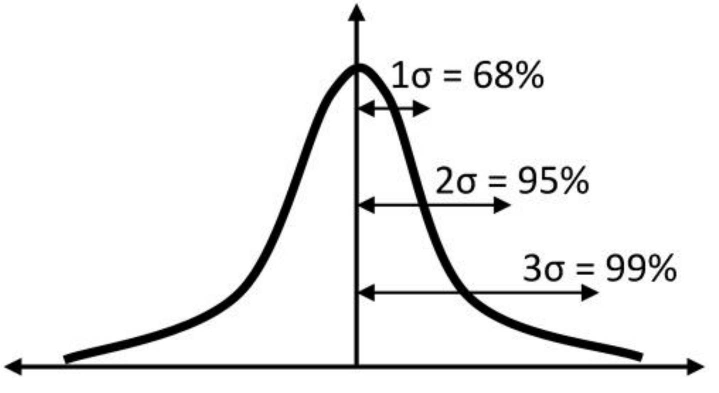

- When $\delta = \Omega(n^{1/2})$ lattice problems can be reduced to LWE [Regev09]

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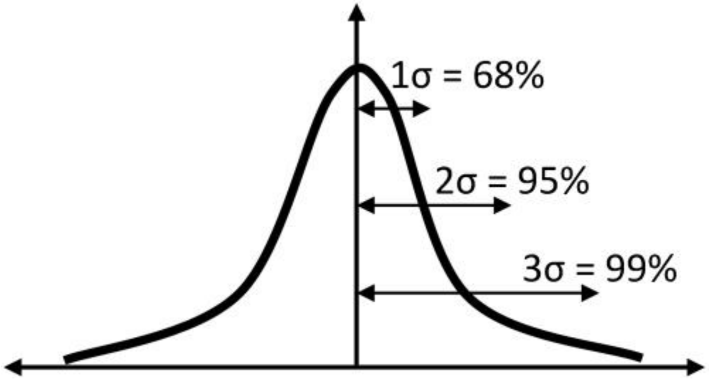
- Cor: When $\delta = o(n^{1/2})$, LWE has a sub-exponential time algorithm

Thm → Cor



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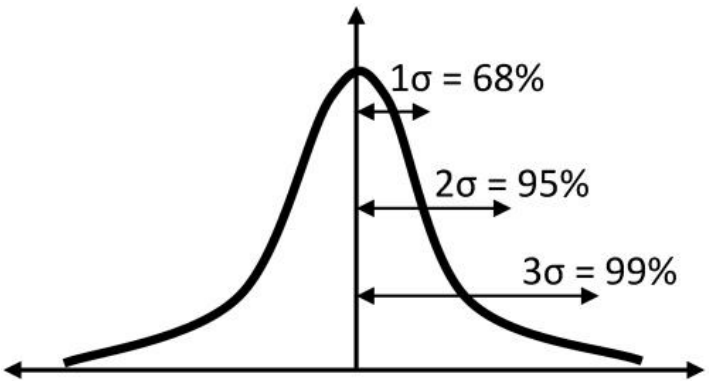
$|c| < K \delta^2$



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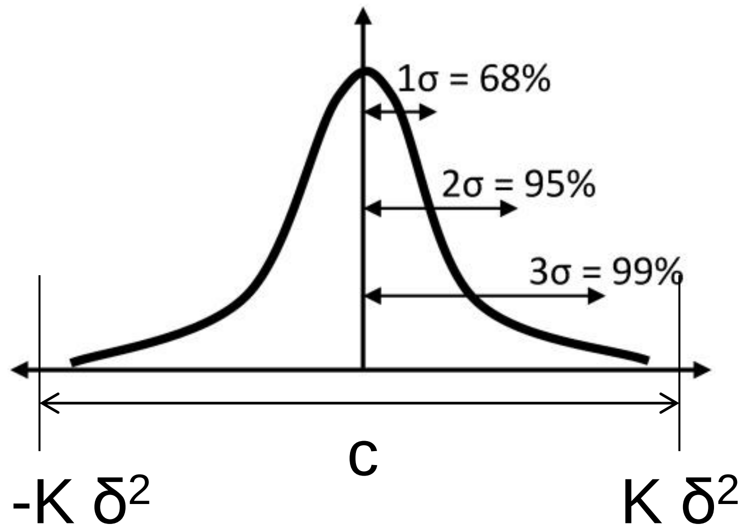

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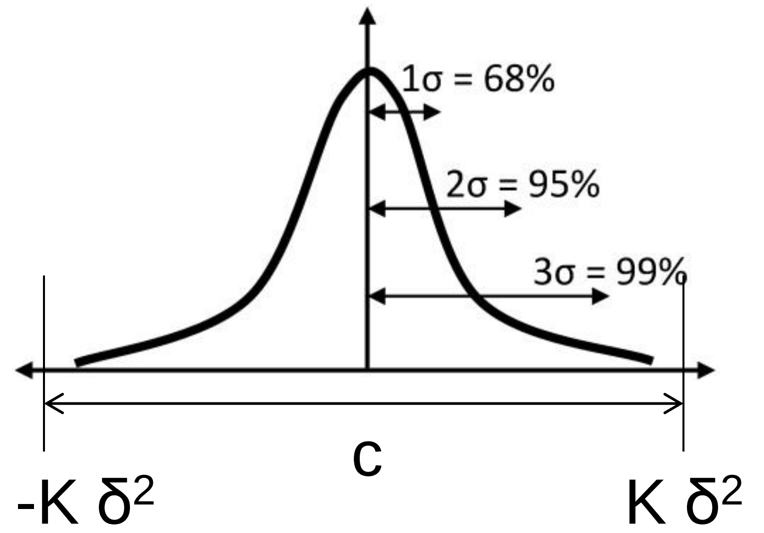

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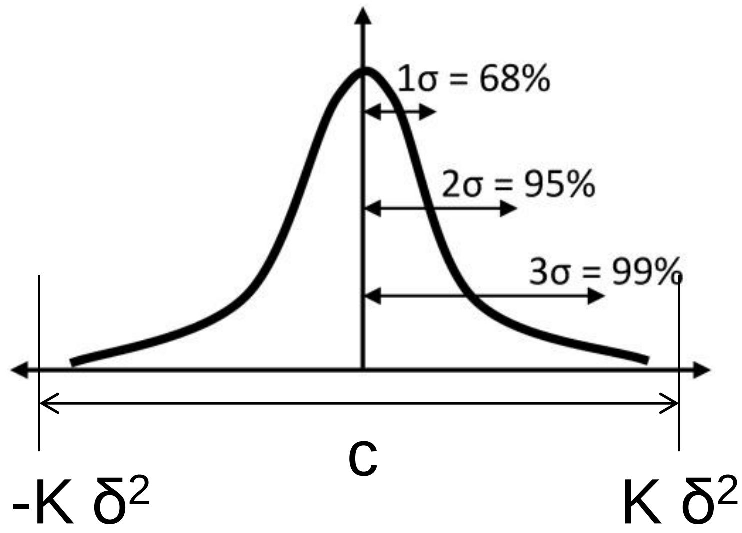

$K = 100 \log n$

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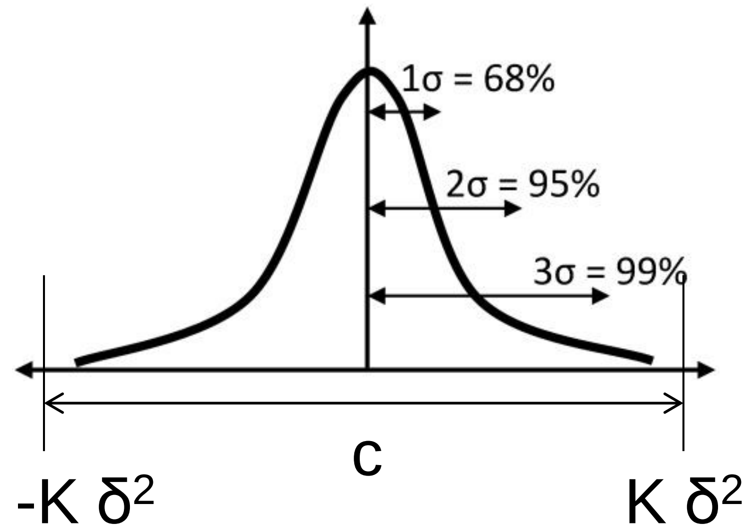

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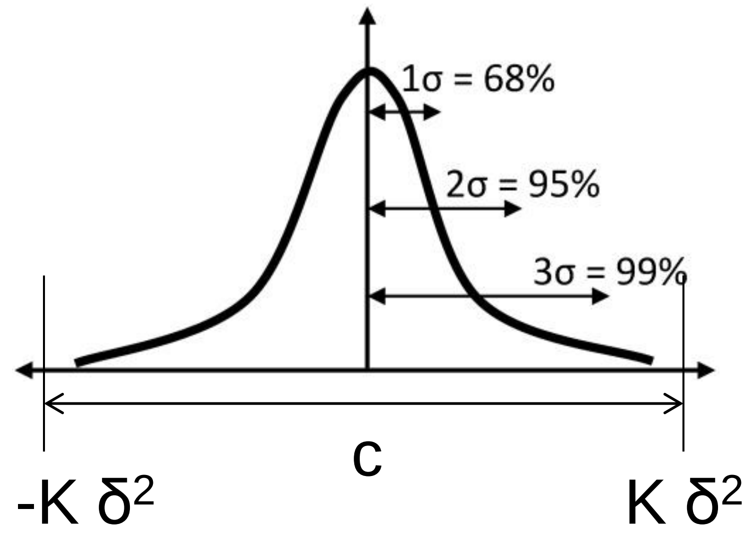

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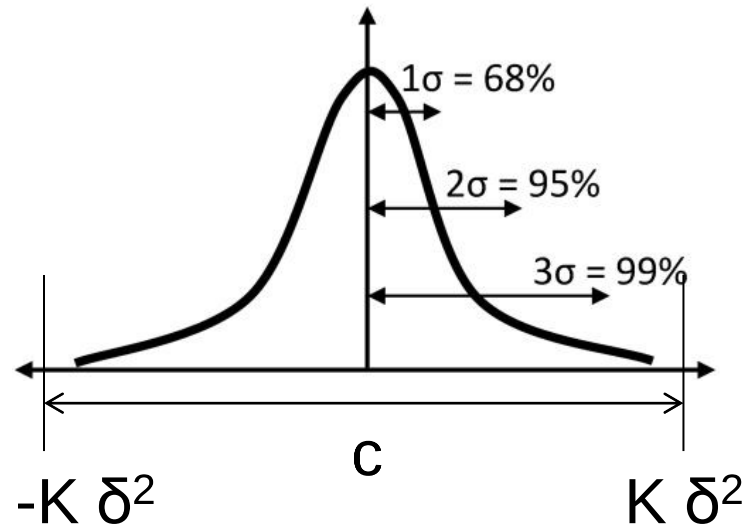

$K = 100 \log n$

Negligible difference between LWE and LWSE,
Algorithm still success with high probability

Open Problems

- Non-trivial algorithm for the original model using linearization
- Possible lower bound for special kind of linear equation systems
- Improve the algorithm for learning with errors?

Thank You

Questions?