Efficient Cryptography from Generalized Compact Knapsacks

Vadim Lyubashevsky
INRIA & ENS Paris

The Knapsack Problem

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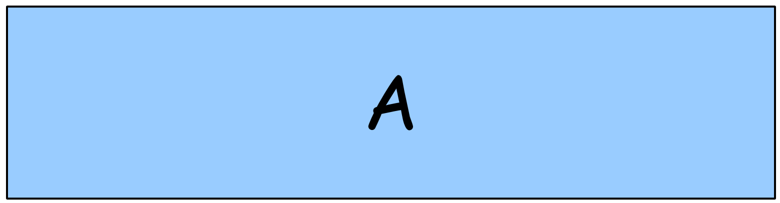

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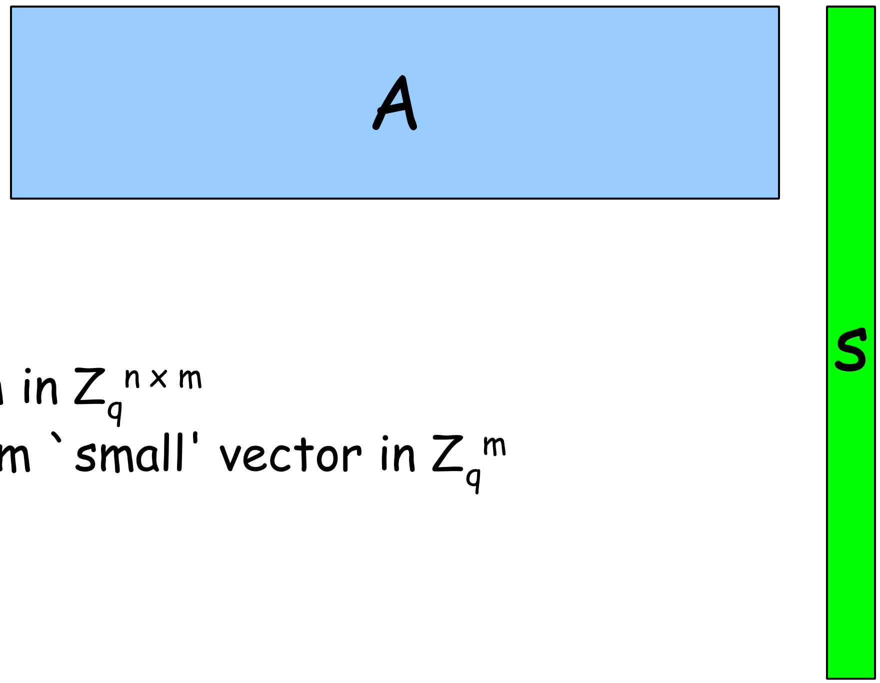

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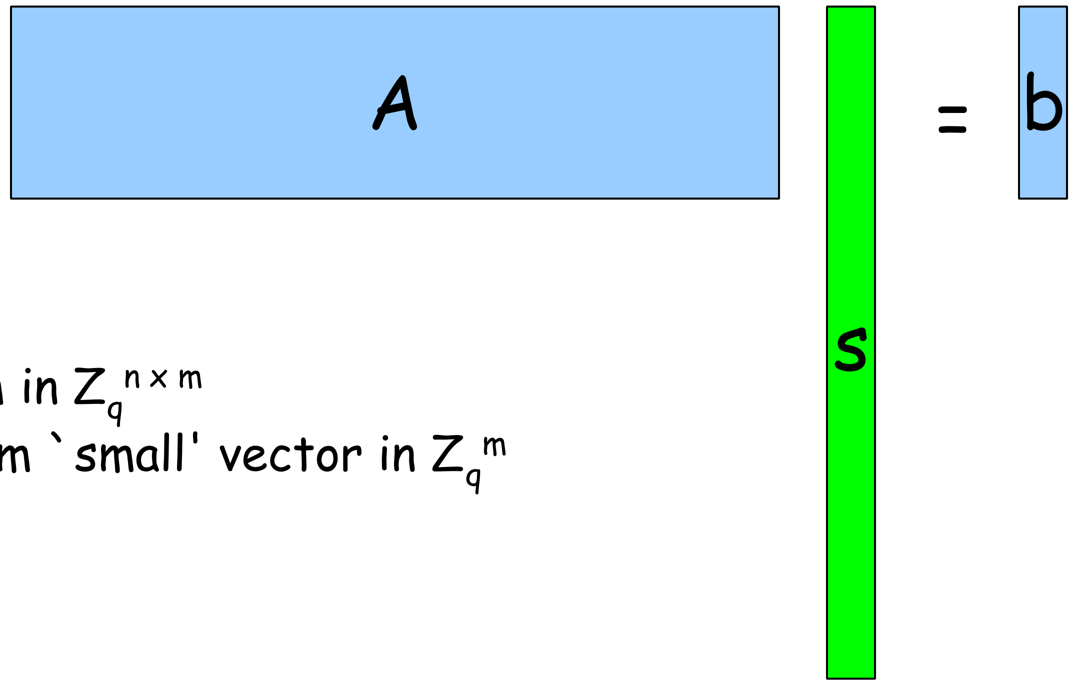

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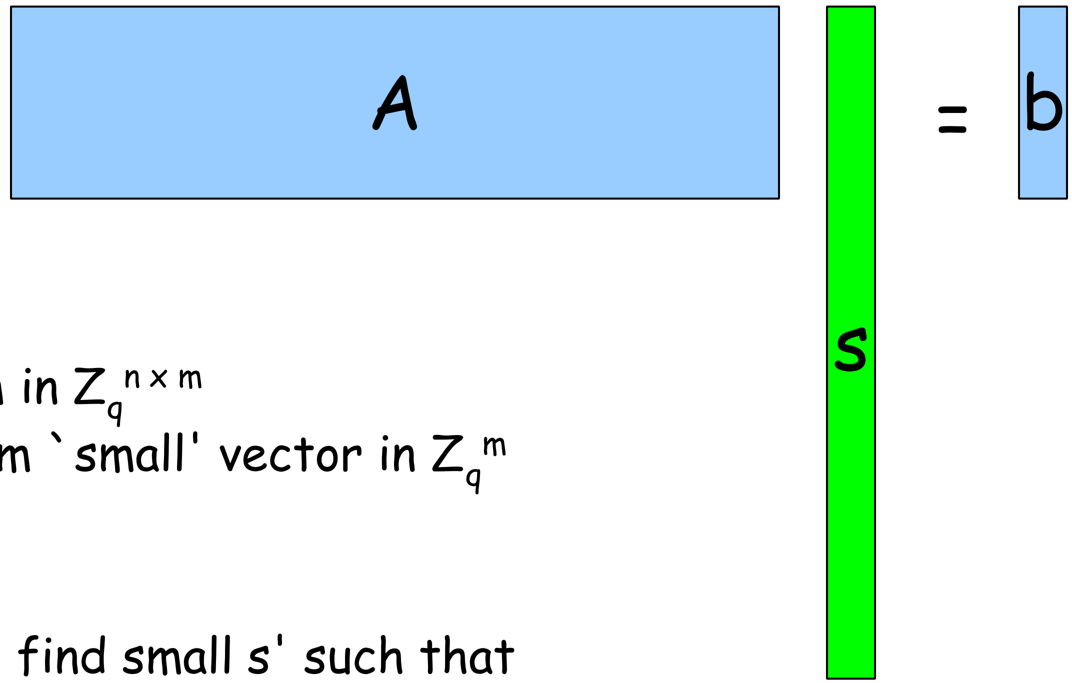

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s is a random `small' vector in Z_q^m

b=As mod q

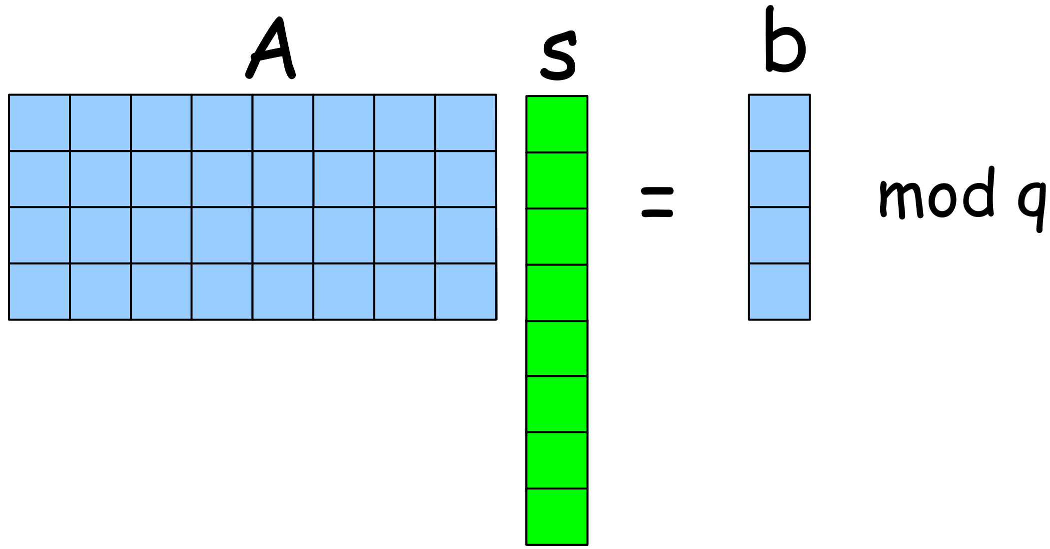
Given (A,b), find small s' such that

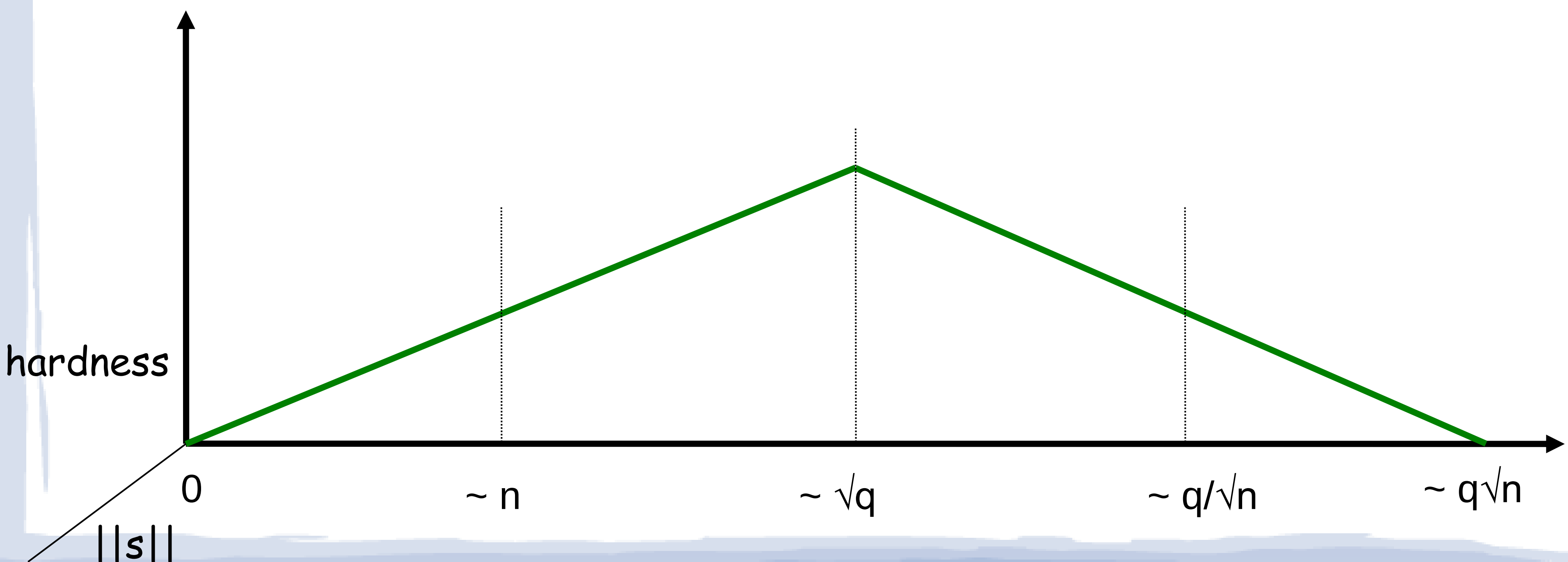
As'=b mod q

The Knapsack Problem

$$\overset{A}{\begin{bmatrix} 4 & 11 & 6 & 8 & 1 & 0 & 0 & 0 \\ 7 & 7 & 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 9 & 12 & 5 & 0 & 0 & 1 & 0 \\ 1 & 3 & 14 & 9 & 0 & 0 & 0 & 1 \end{bmatrix}} \overset{s}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 12 \\ 10 \\ 8 \\ 10 \end{bmatrix}} \quad \text{mod } 17$$

Hardness of the Knapsack Problem





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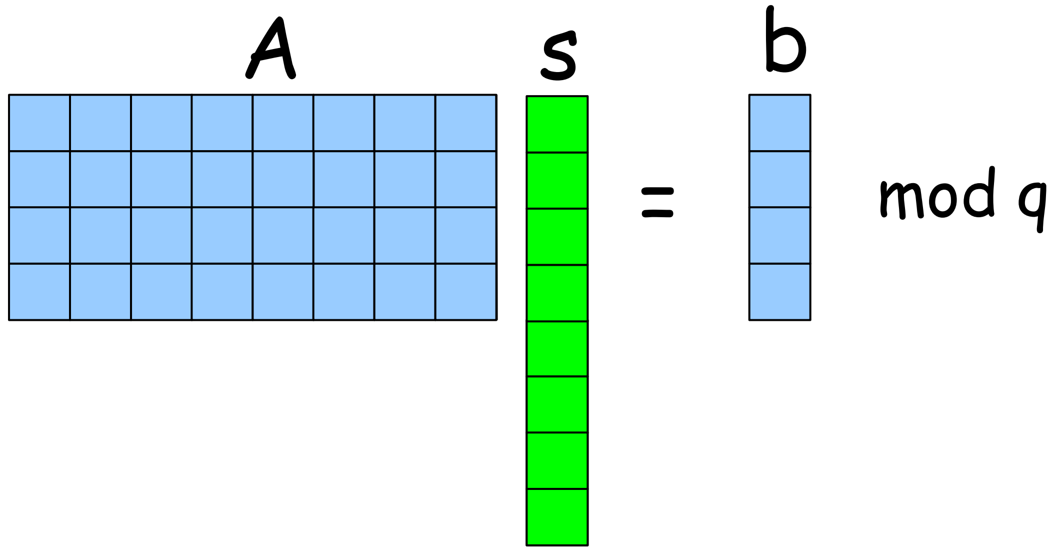

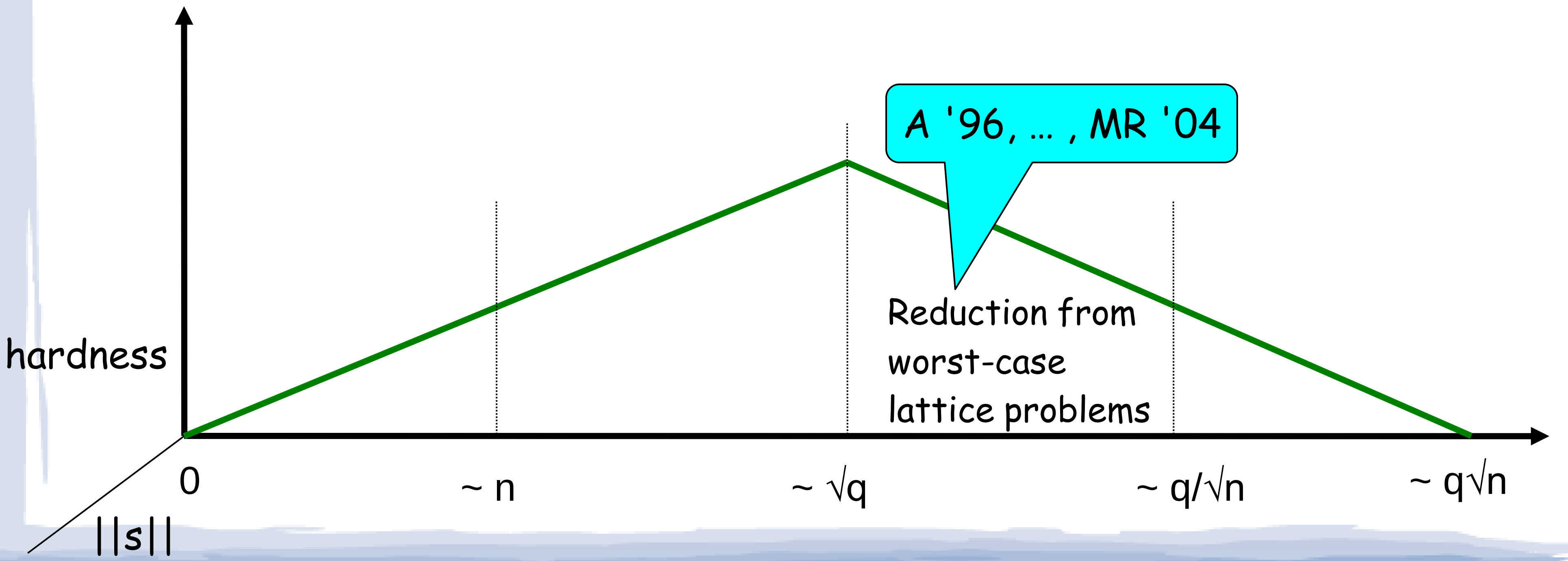

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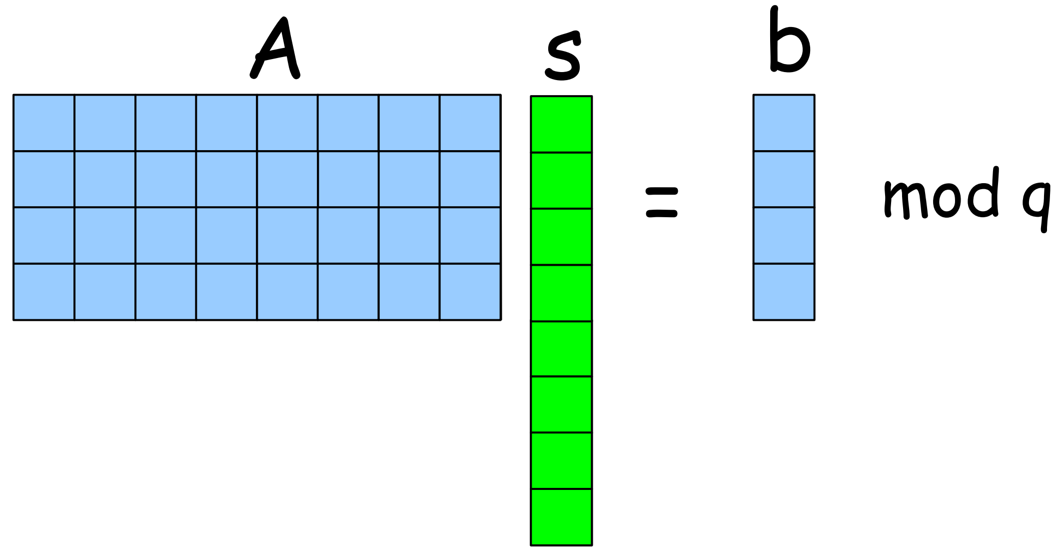
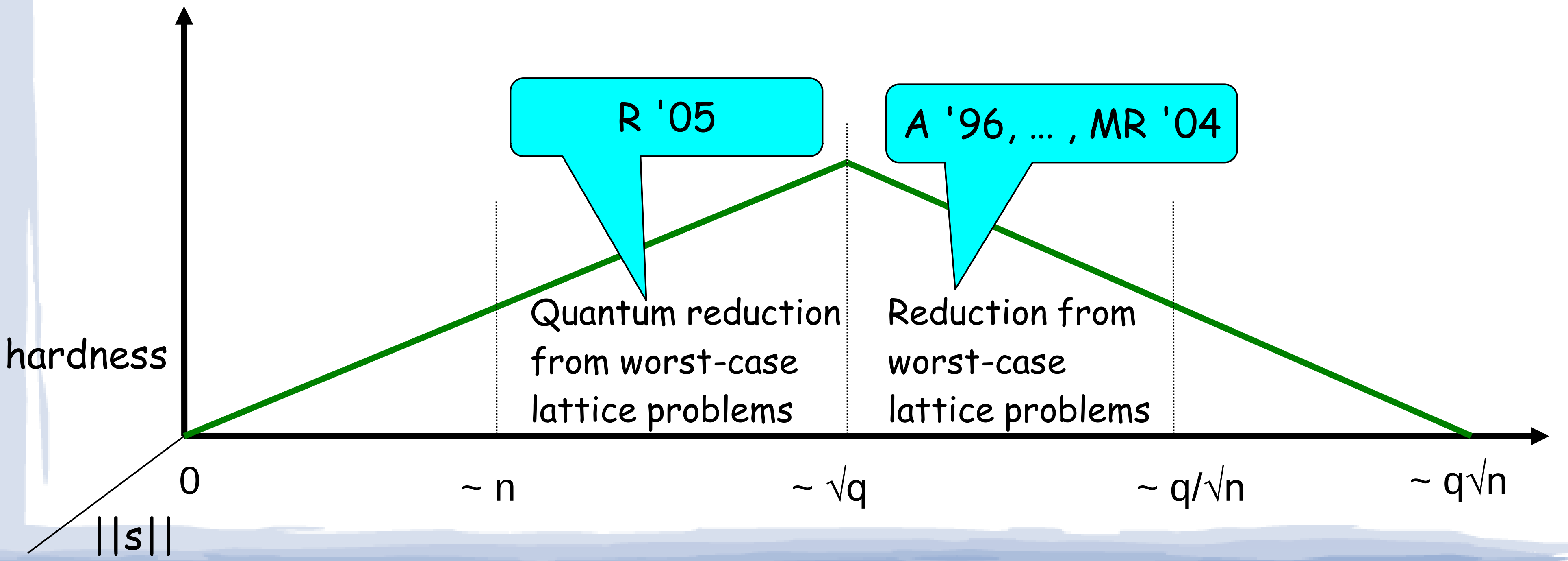
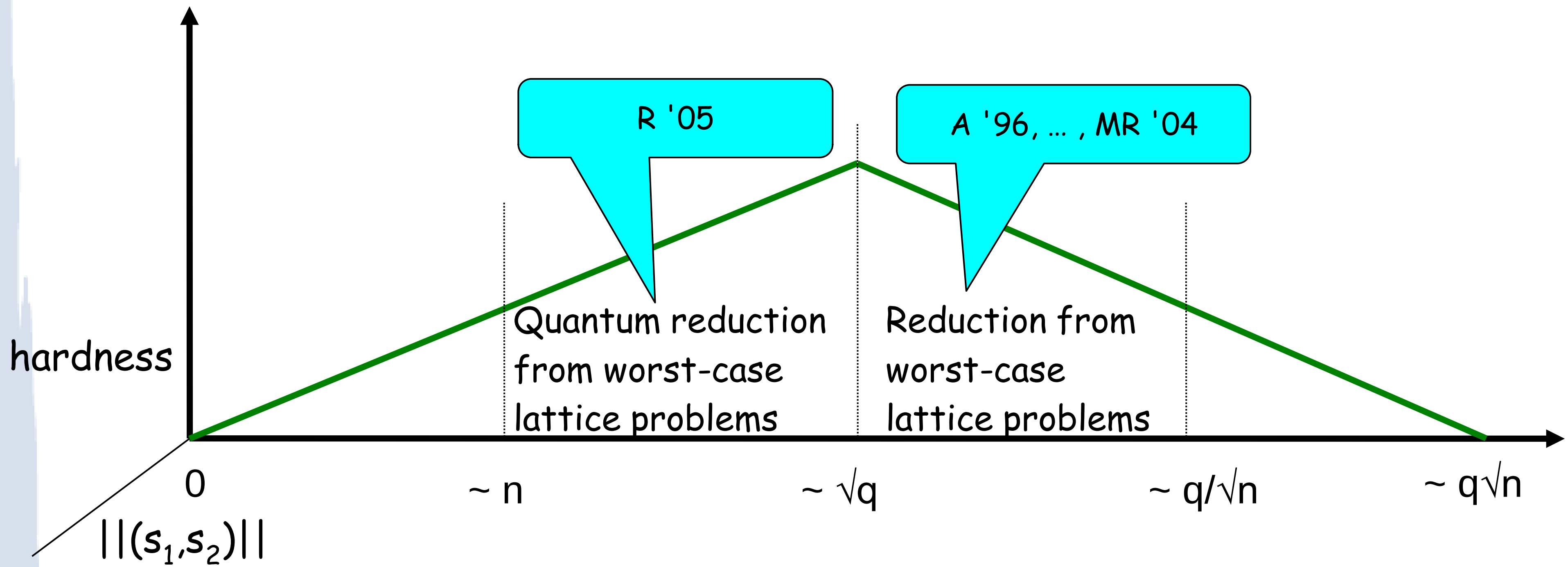




Cryptographic Primitives



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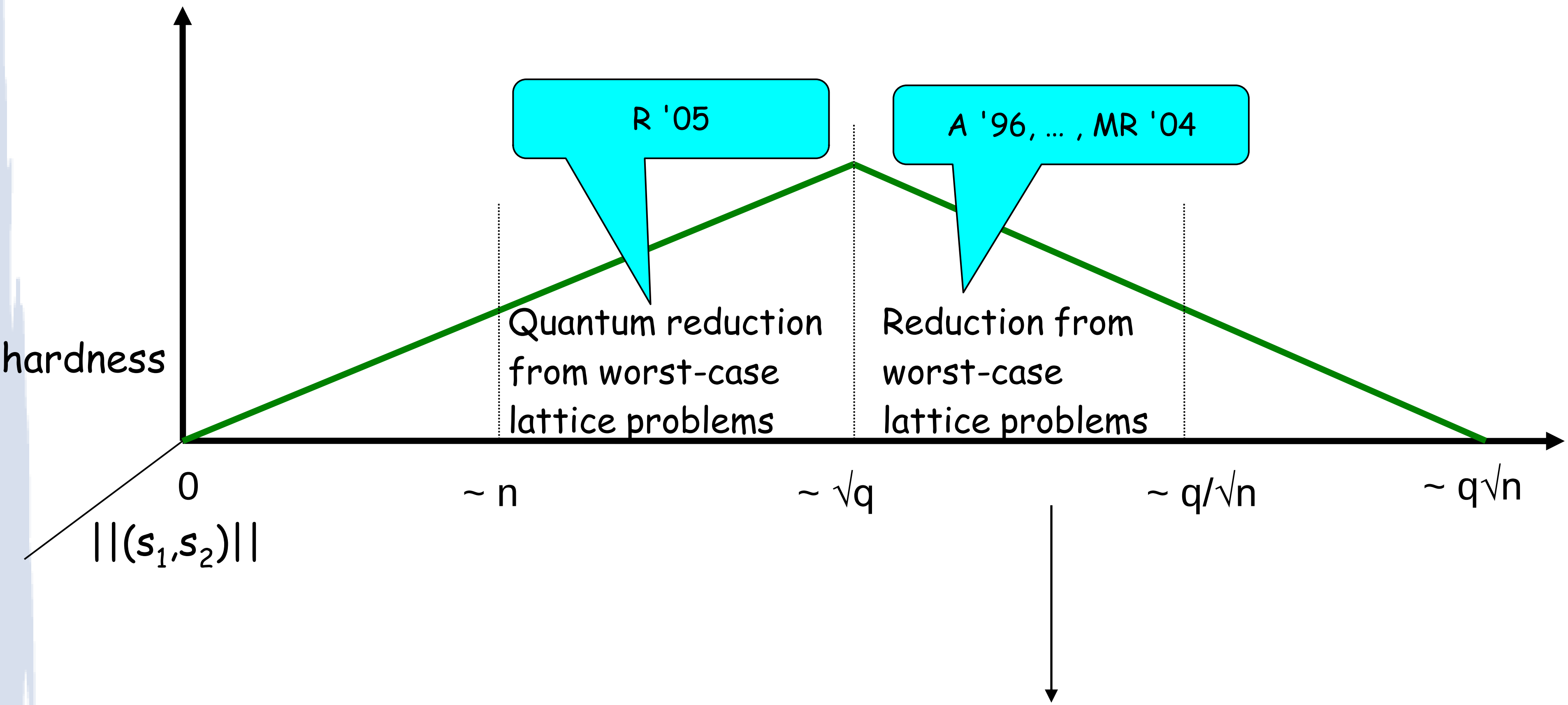

- One-way functions
- Collision resistant hash functions
- Identification schemes
- Digital signatures

(minicrypt)

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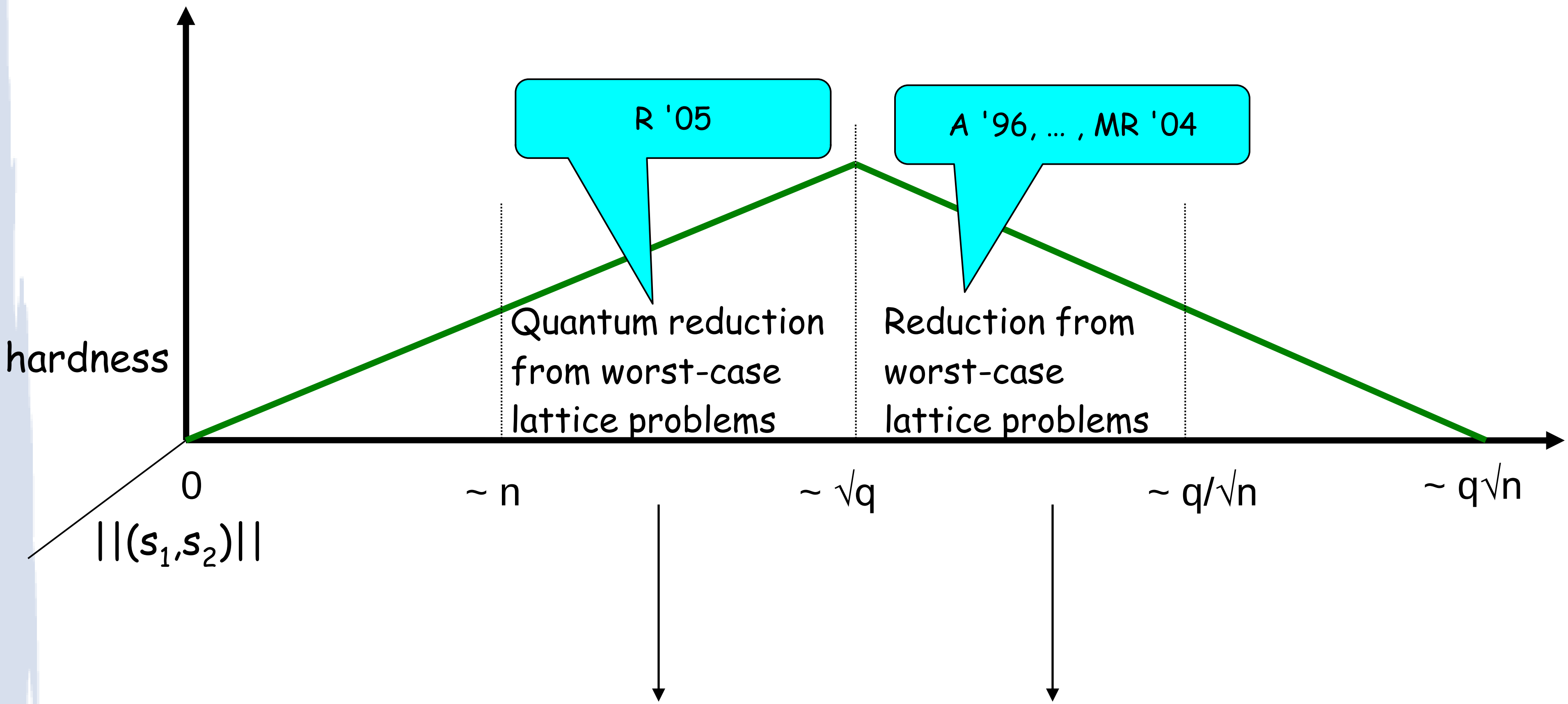

- Public-Key Encryption
- Identity-Based Encryption

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(cryptomania)

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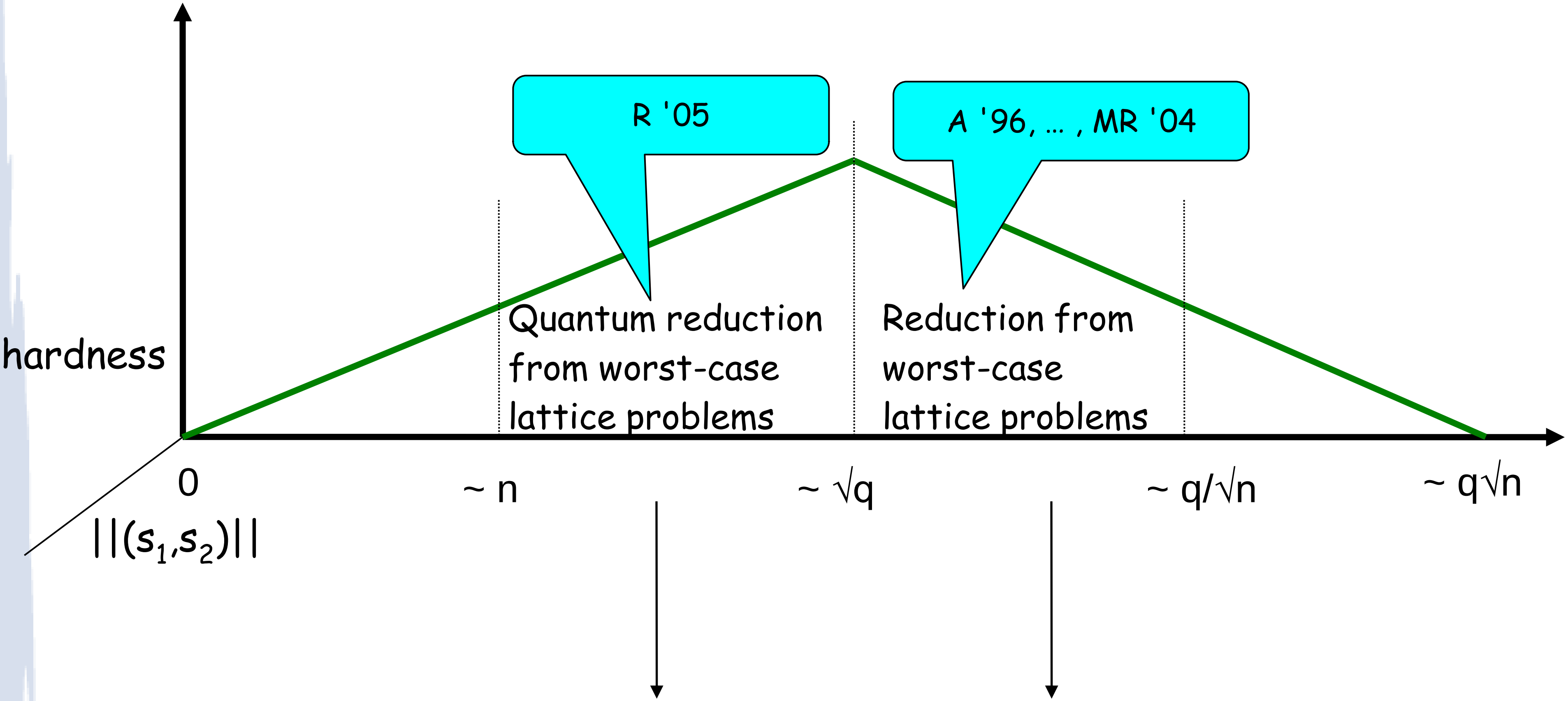
Practical Cryptographic Primitives?



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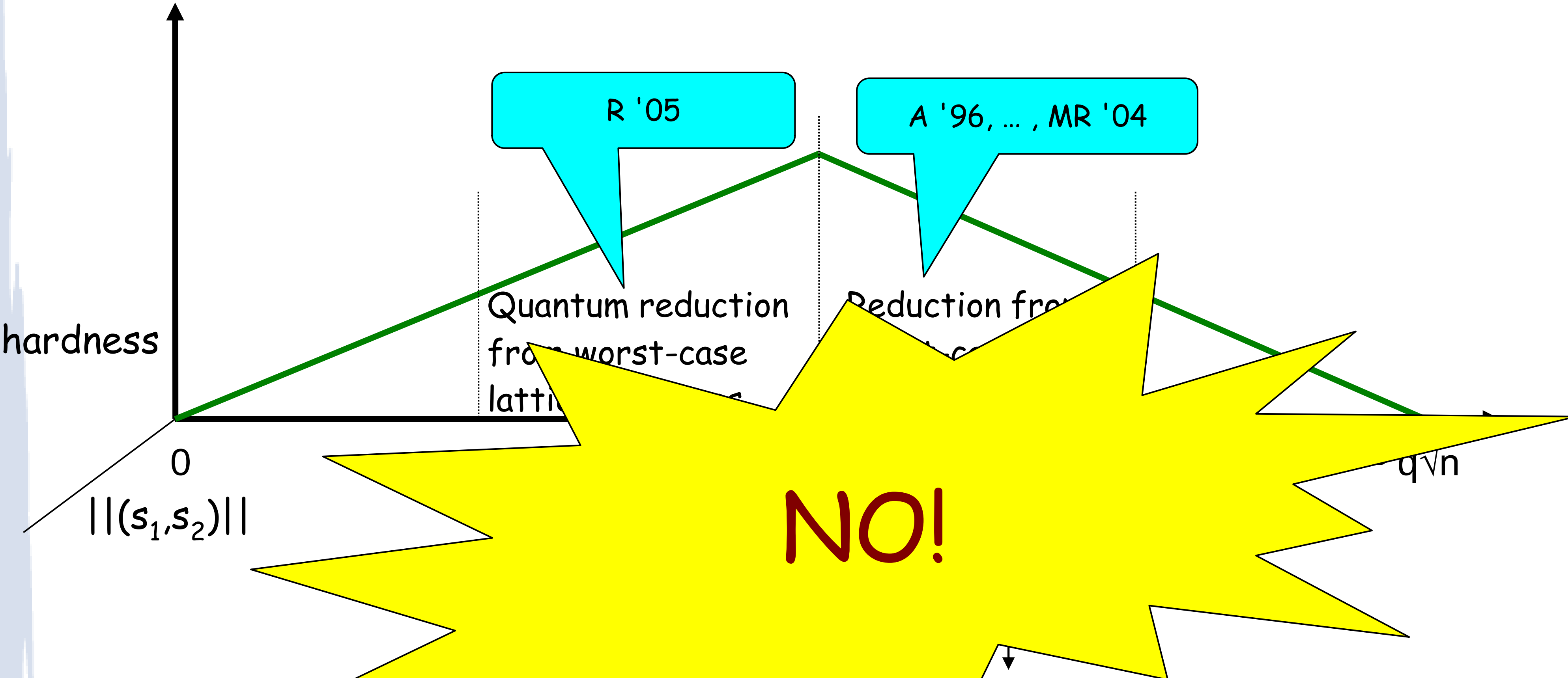

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Can we have the same properties and practicality?

The Compact Knapsack Problem

$$\overset{A}{\begin{bmatrix} 4 & -1 & -2 & -7 & 1 & 0 & 0 & 0 \\ 7 & 4 & -1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 7 & 4 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 7 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}} \overset{s}{\begin{bmatrix} \ \\ \ \\ \ \\ \ \\ \ \\ \ \\ \ \\ \ \end{bmatrix}} = \overset{b}{\begin{bmatrix} \ \\ \ \\ \ \\ \ \end{bmatrix}} \mod q$$

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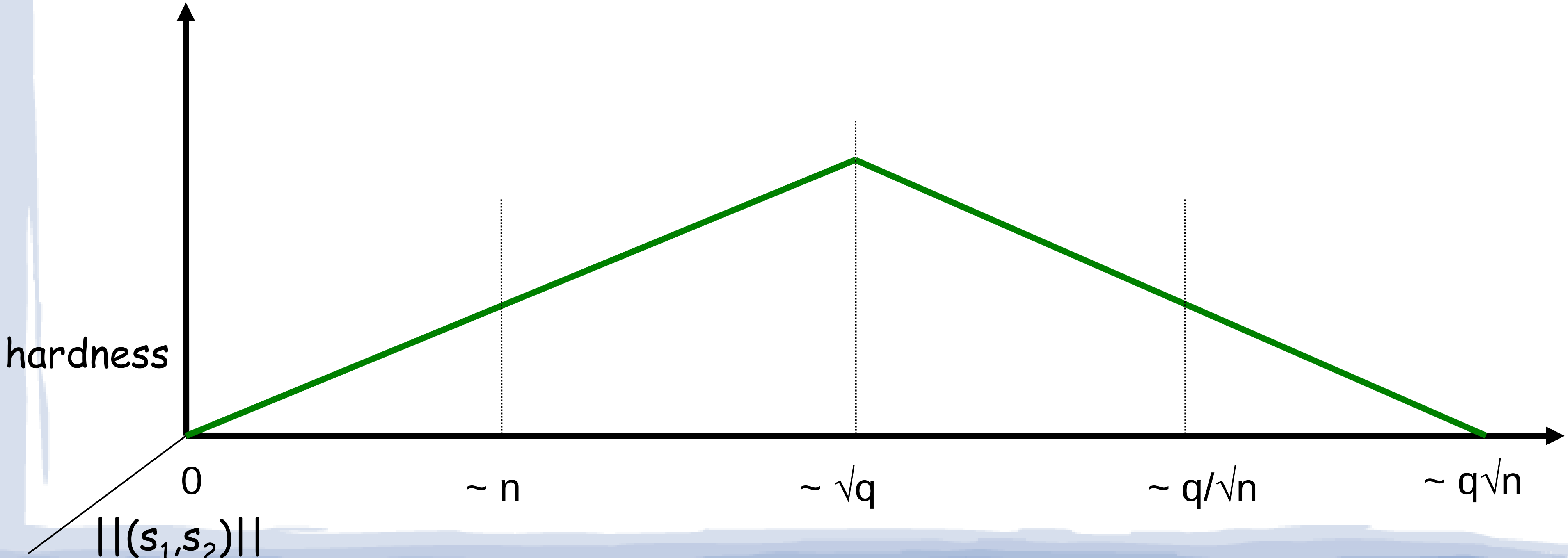
Equivalent to polynomial multiplication in the ring $R = Z_q[x]/(x^n+1)$

$$as_1 + s_2 = b$$

Hardness of the Compact Knapsack Problem

$$as_1 + s_2 = b \mod q$$



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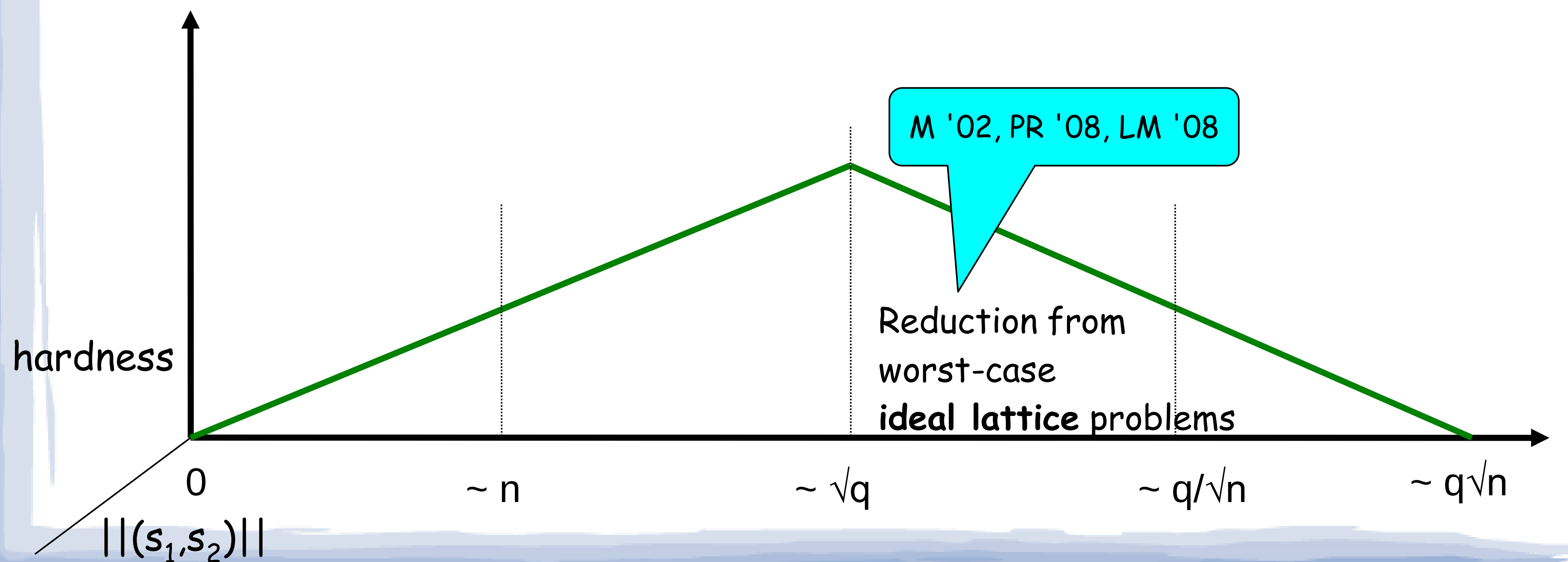

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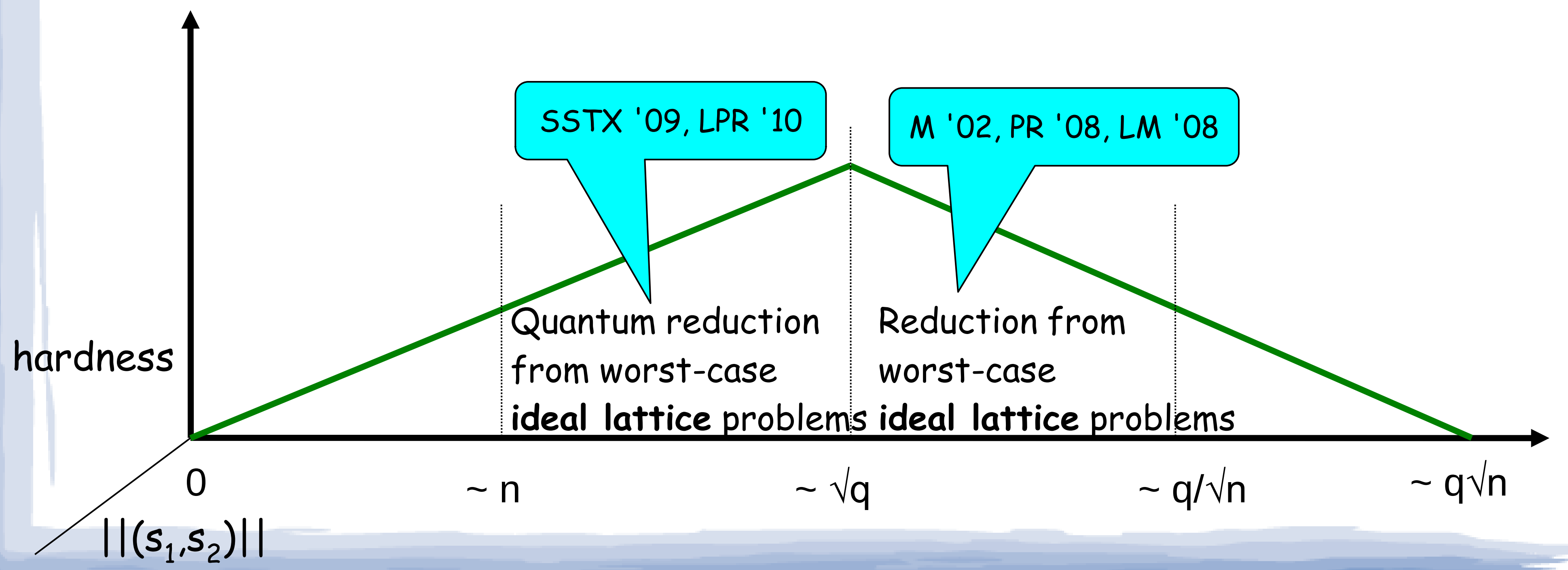

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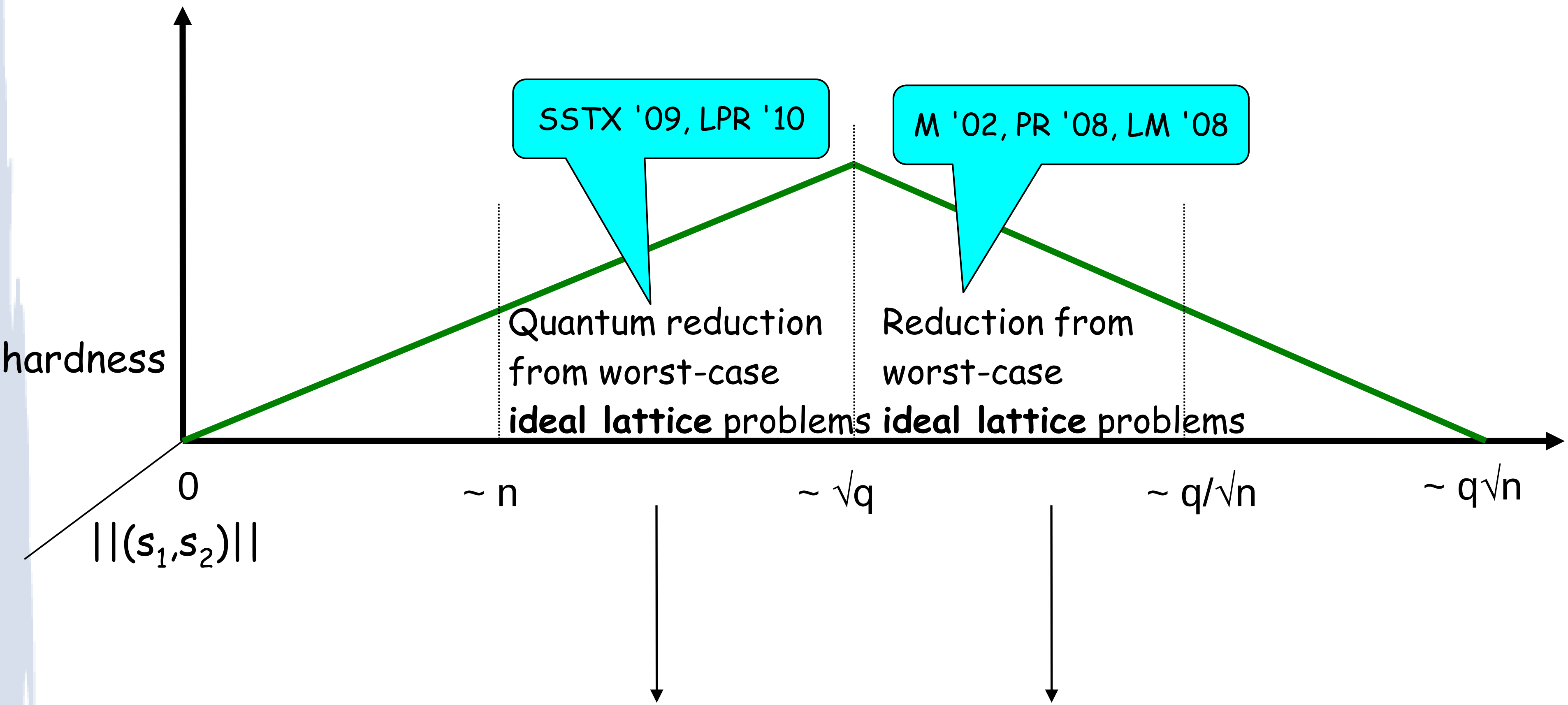

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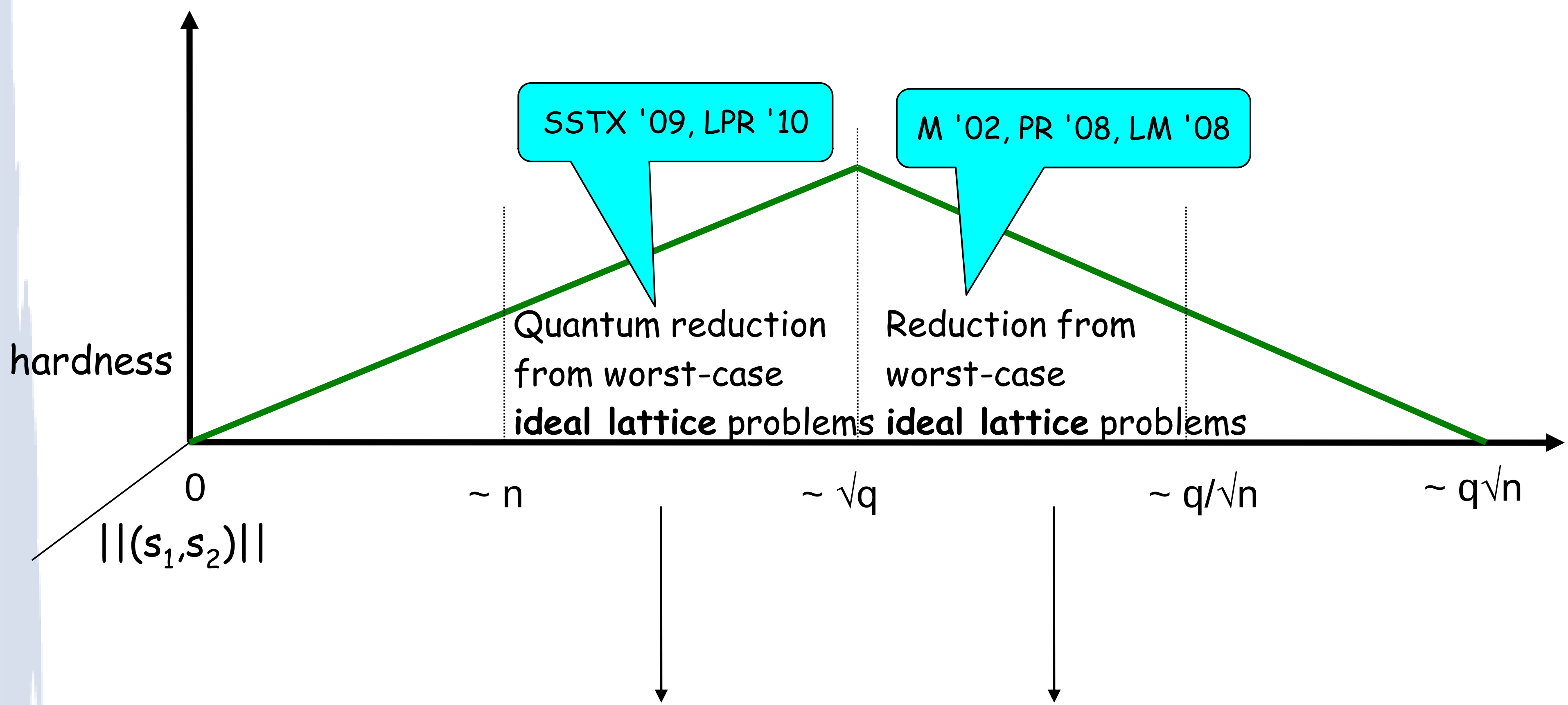

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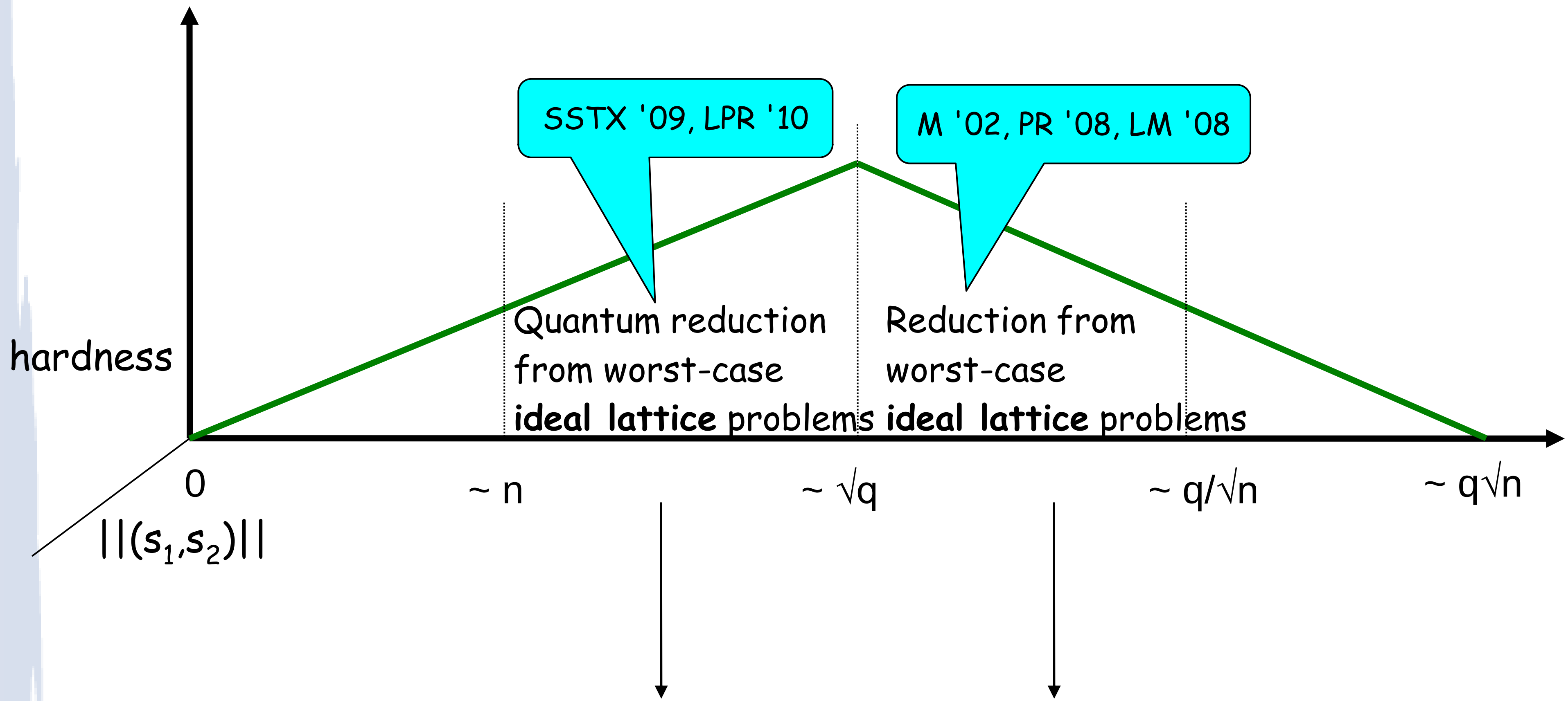

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- If we want lattices to be a viable alternative, we **must** make signatures smaller
  - In my opinion, this, and constructing 'practical' fully-homomorphic encryption are the two most important problems in lattice-based crypto

In this Talk

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- A new way to construct lattice-based signature schemes

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- A new way to construct lattice-based signature schemes
- For ~ 80 bits of security:
  - public key ~ 12,000 bits
  - secret key ~ 1700 bits
  - signature size ~ 9000 bits
  - much faster than RSA/EC signatures

Digital Signature Schemes

Consist of three algorithms: Key-Generate, Sign, and Verify

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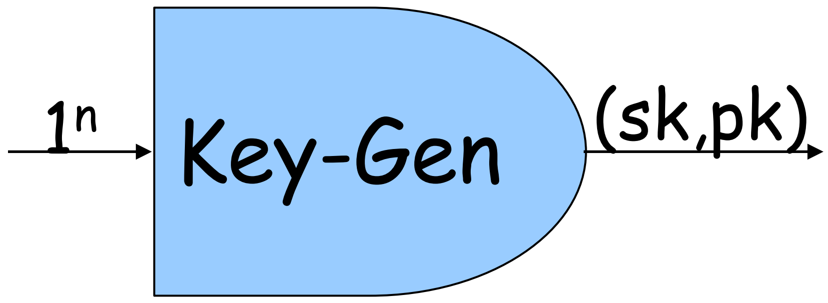

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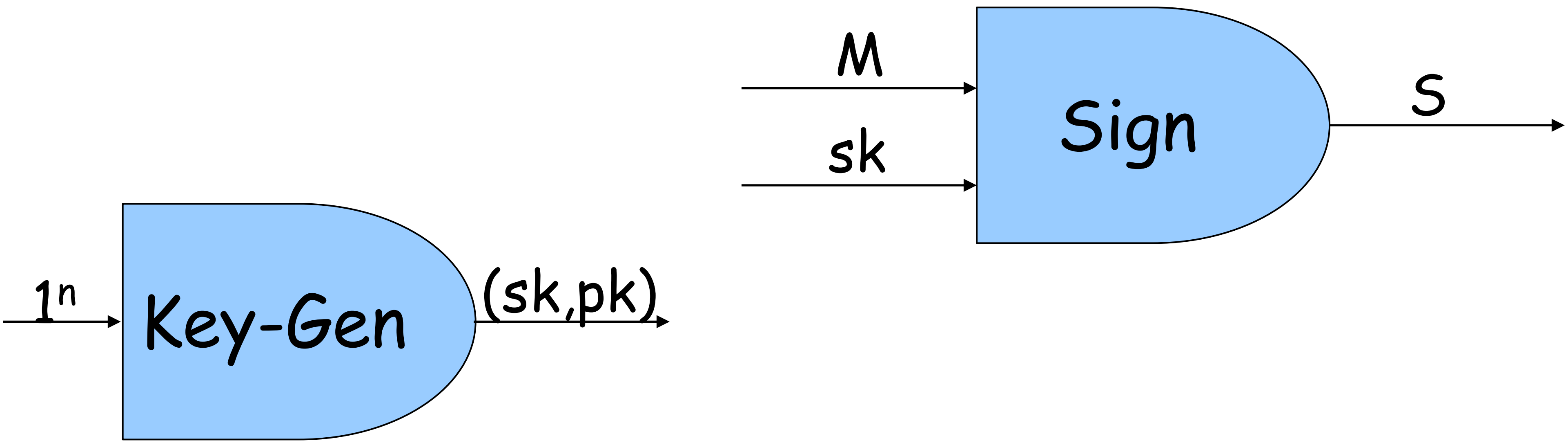

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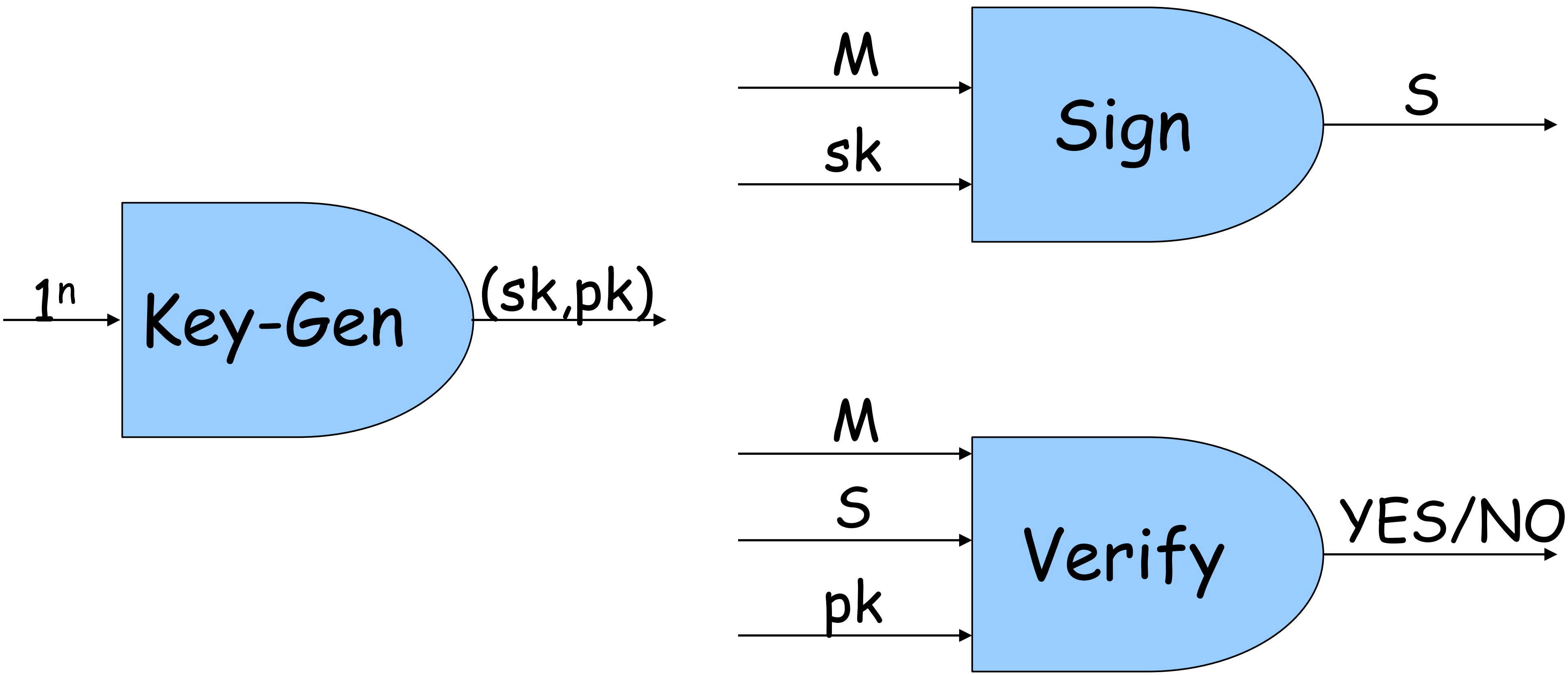

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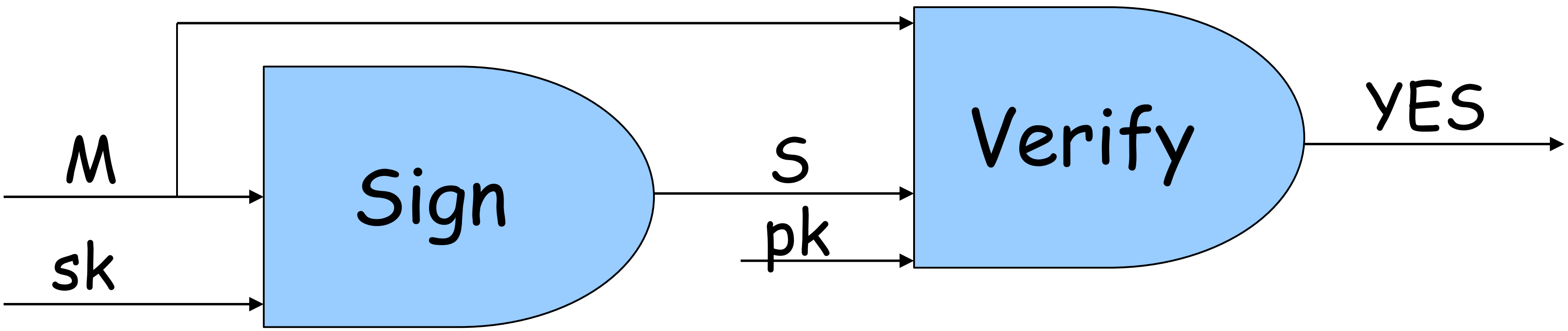

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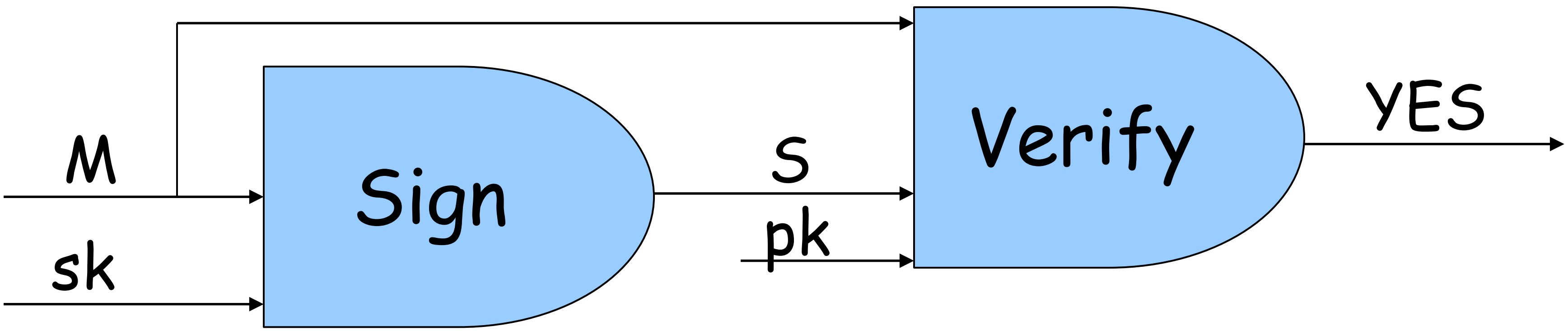
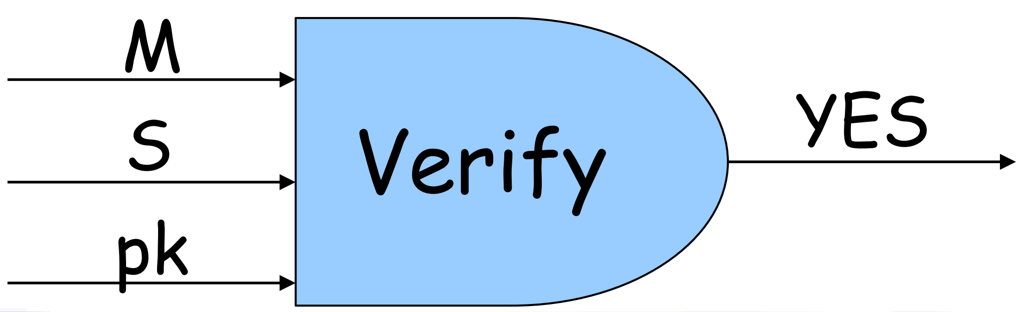


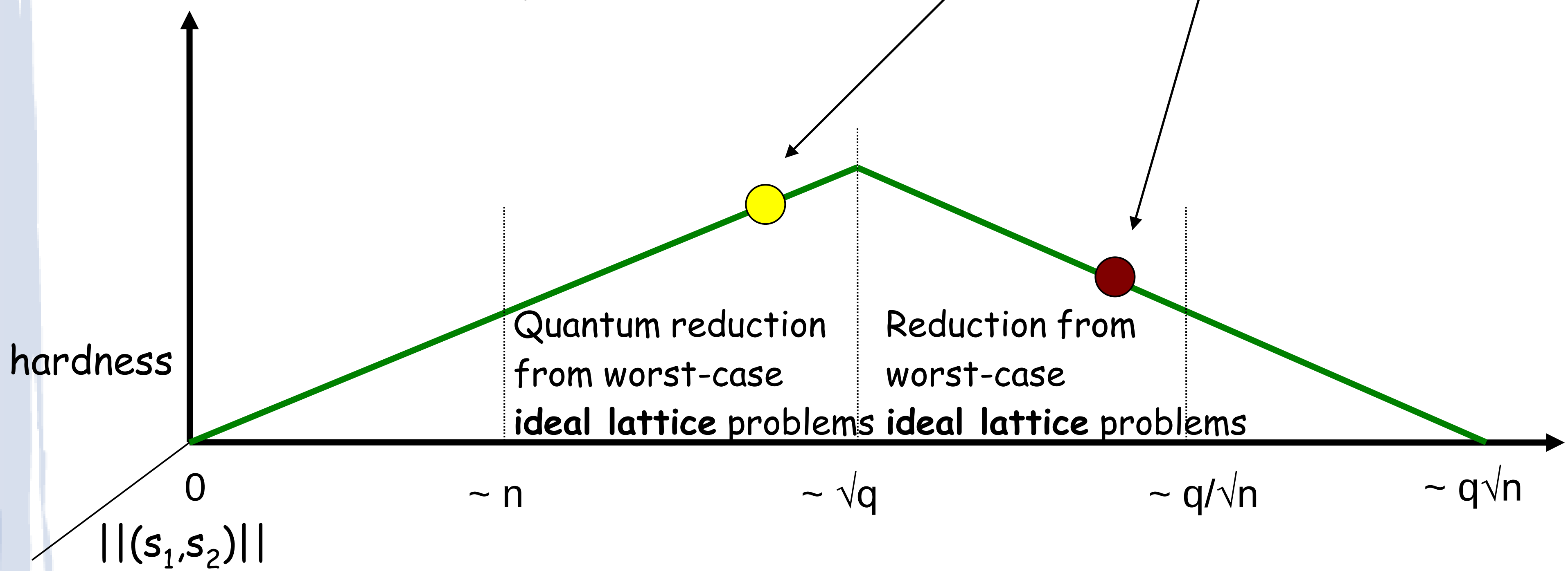
2. Security

Unless M has been signed, cannot find an S such that



Super High-Level Idea Behind the New Construction

Is it better to have a scheme based on this problem or this problem?
(assuming all other parameters are equal)



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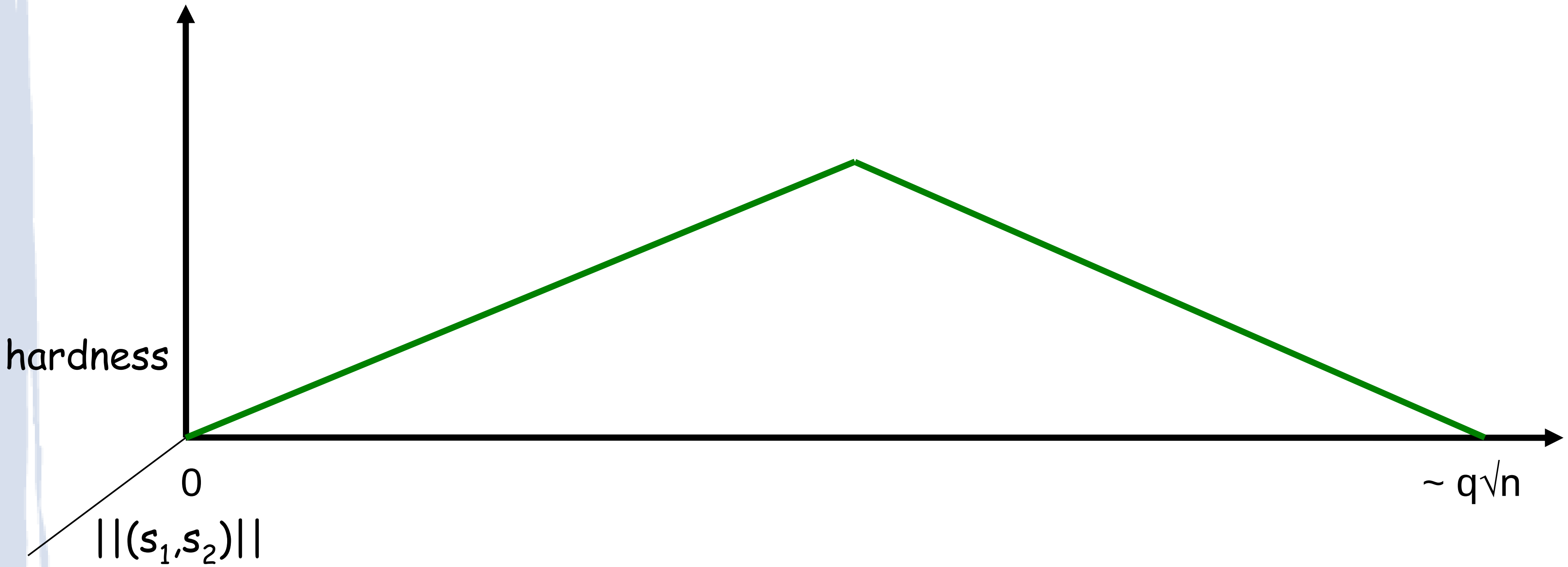

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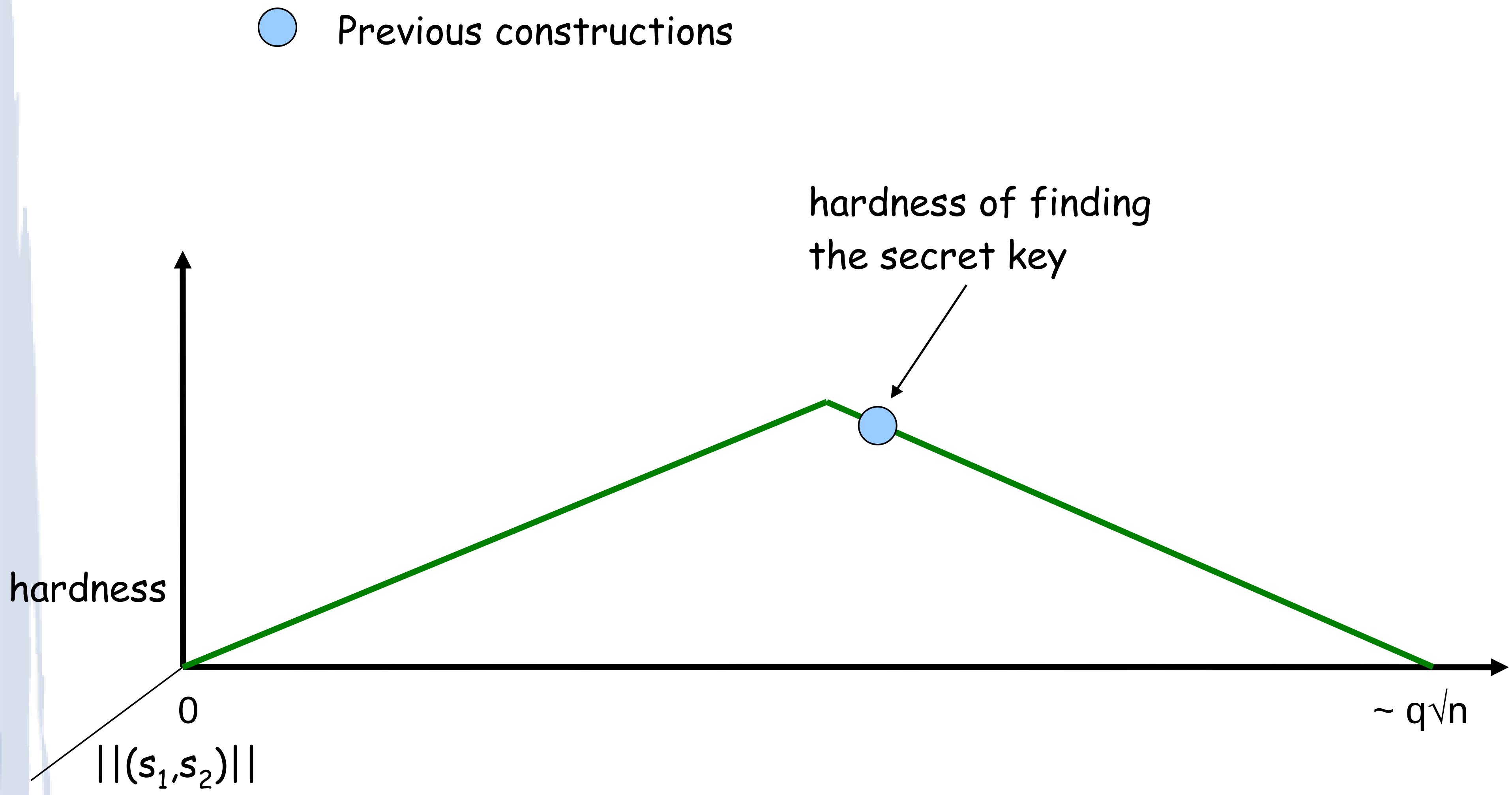

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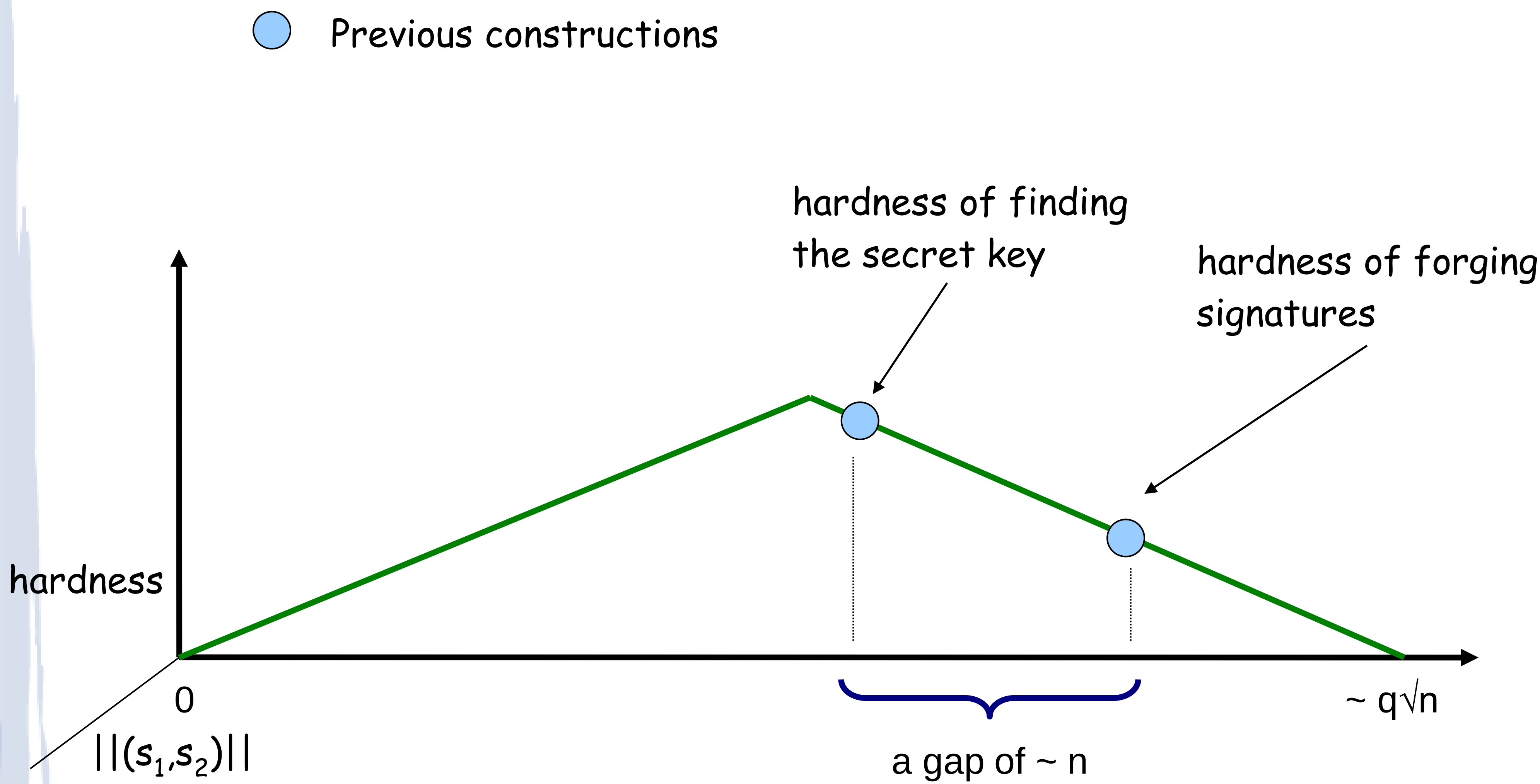

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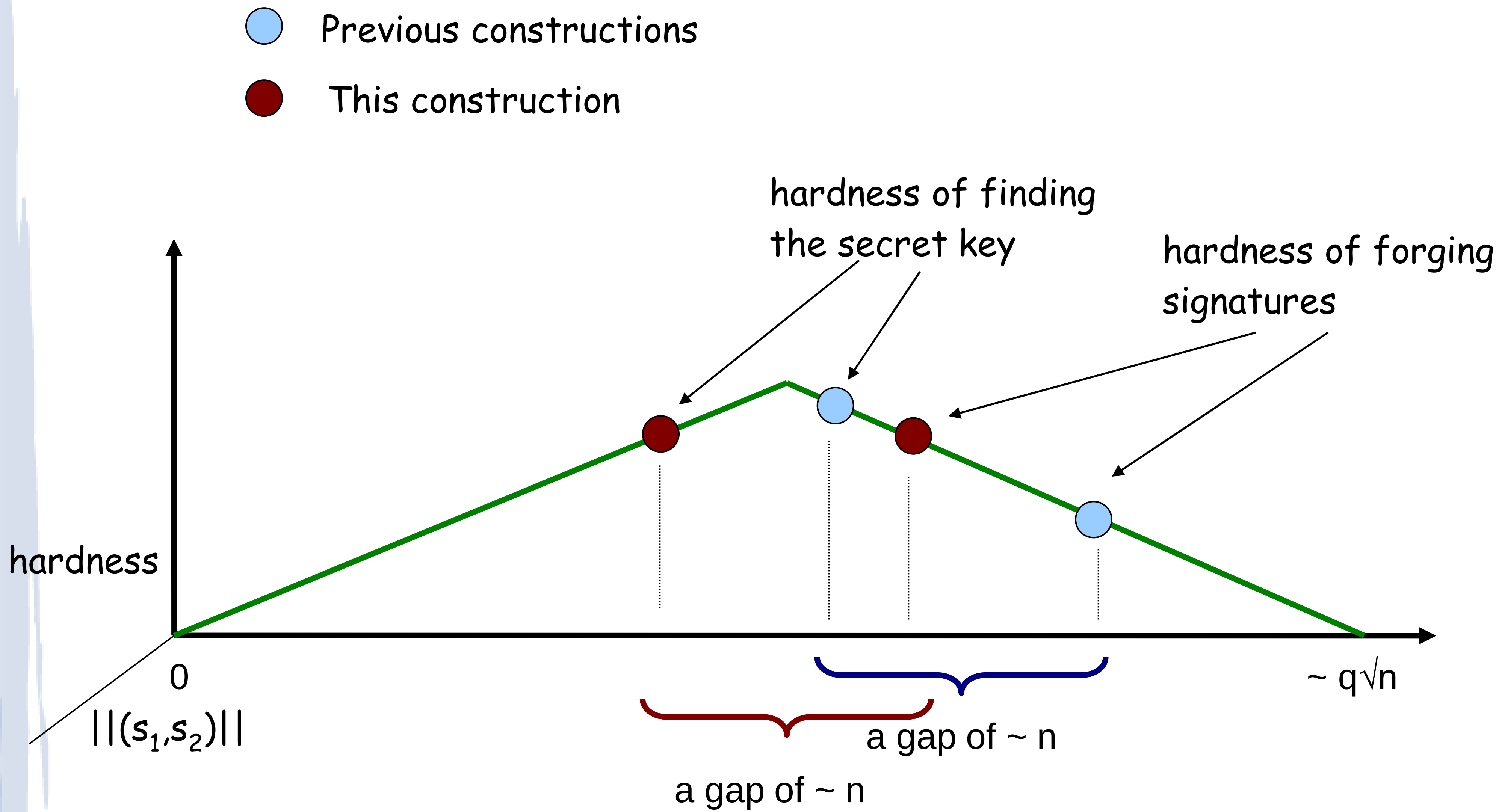

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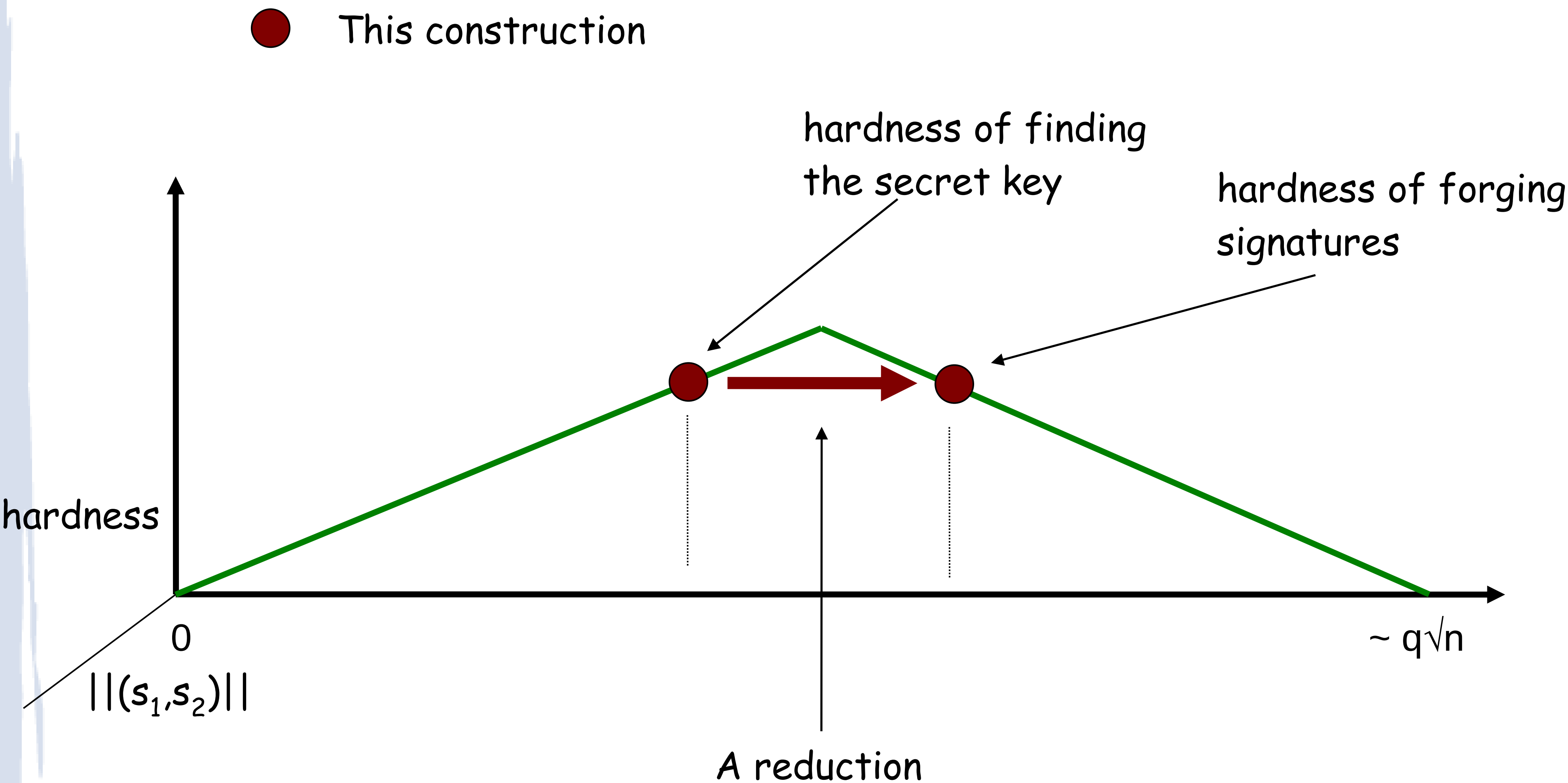

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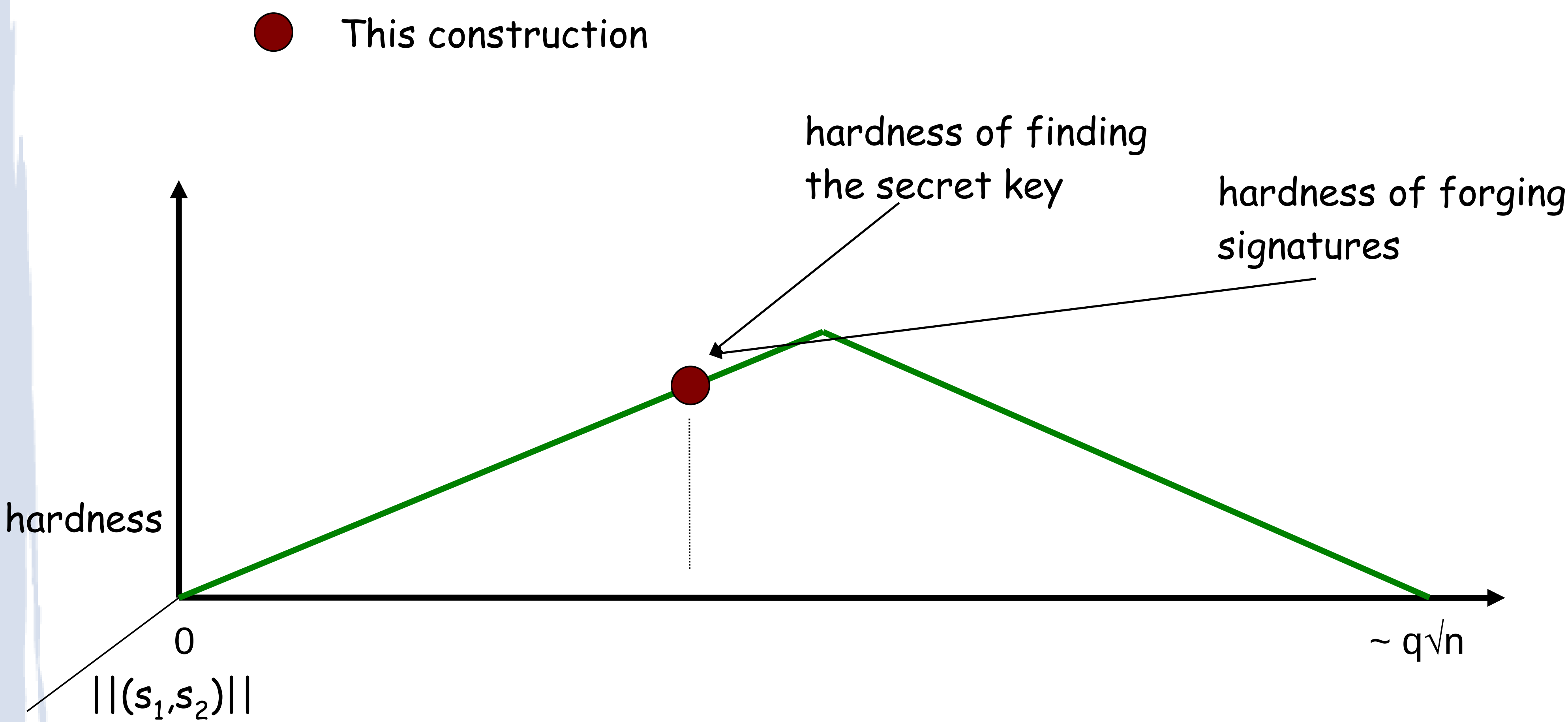

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  q is a prime ($q = 1 \bmod 2n$)

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- R_k = { polynomials in R with coefficients in the range $[-k,k]$ }

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(The Search Version)

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SCK(k):

- pick random a in R
- pick random s_1, s_2 in R_k
- output (a, $b = as_1 + s_2$)

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Given (a,b), find s_1, s_2 in R_k such that $as_1+s_2 = b$

(note: there could be more than one solution)

The Compact Knapsack Problem

(The Decision Version)

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- pick random c in {0,1}
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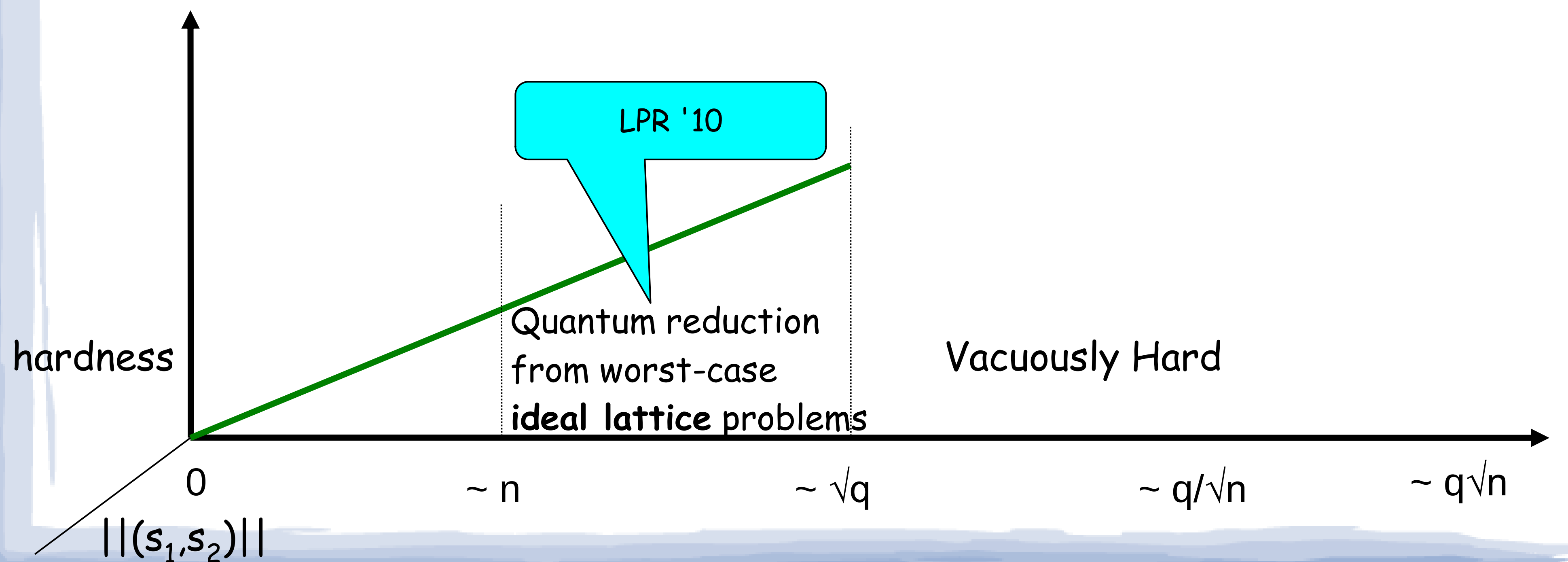
- output (a, $b=as_1 + s_2 + cu$)

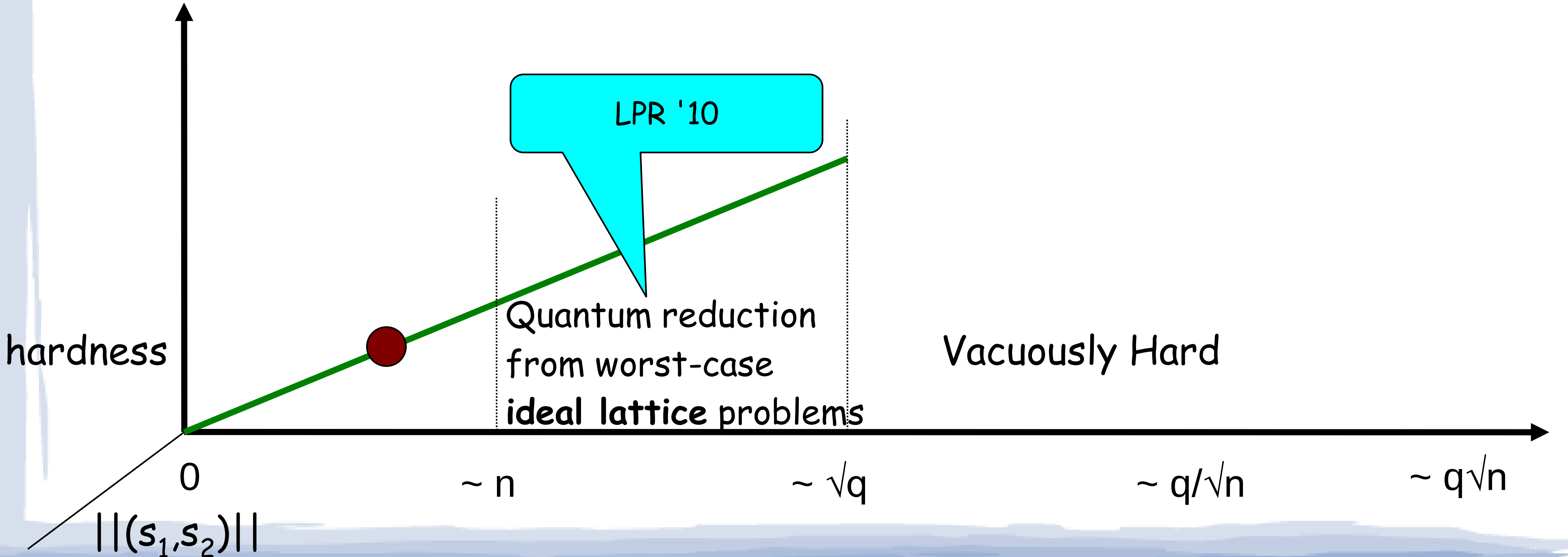
Given (a,b), find c (be correct with probability > 1/2)

- Note: if k is too big, the problem is vacuously hard

Hardness of the Compact Knapsack Problem (Decision Version)



For Added Efficiency …



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check that z_1,z_2 are in R_{k-32} and $c=H(az_1 + z_2 - bc, m)$

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signature size ~ nlog(2k)+nlog(2k)+160

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The Signature Scheme
(improved version)

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sign(m)

H only acts on the "high order bits"

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can "compress" z_2

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signature size ~ $n\log(2k)+2n+160$

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check that z_1,z_2 are in R_{k-32} and $c=H(az_1 + z_2 - bc, m)$

The Signature Scheme

sk: s_1, s_2 in R_1 pk: a in R, $b=as_1+s_2$

sign(m)

1. pick random y_1,y_2 in R_k ($k \sim n$)
2. $c = H(ay_1+y_2 , m)$
3. $z_1=cs_1+y_1$, $z_2=cs_2+y_2$
4. if z_1,z_2 are not in R_{k-32} , go back to step 1
5. output (z_1, z_2, c)

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Security Proof

(High Level Idea)

Security Proof
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Given random a in R

Security Proof
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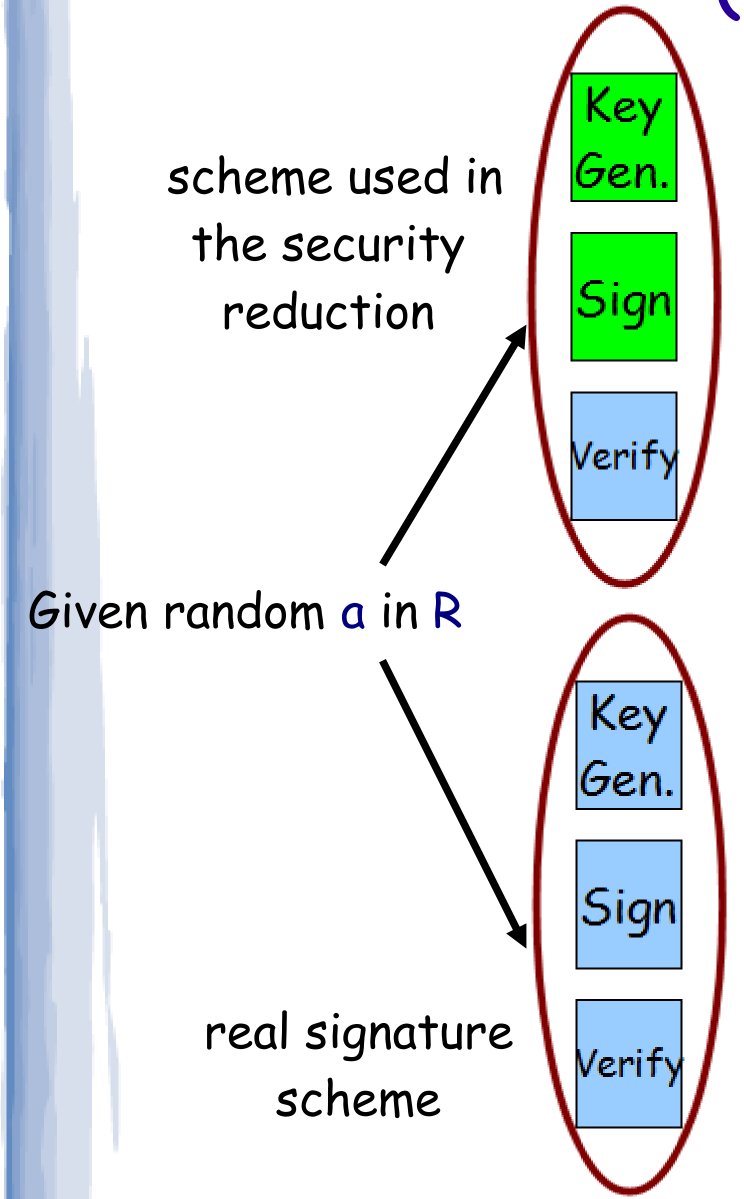

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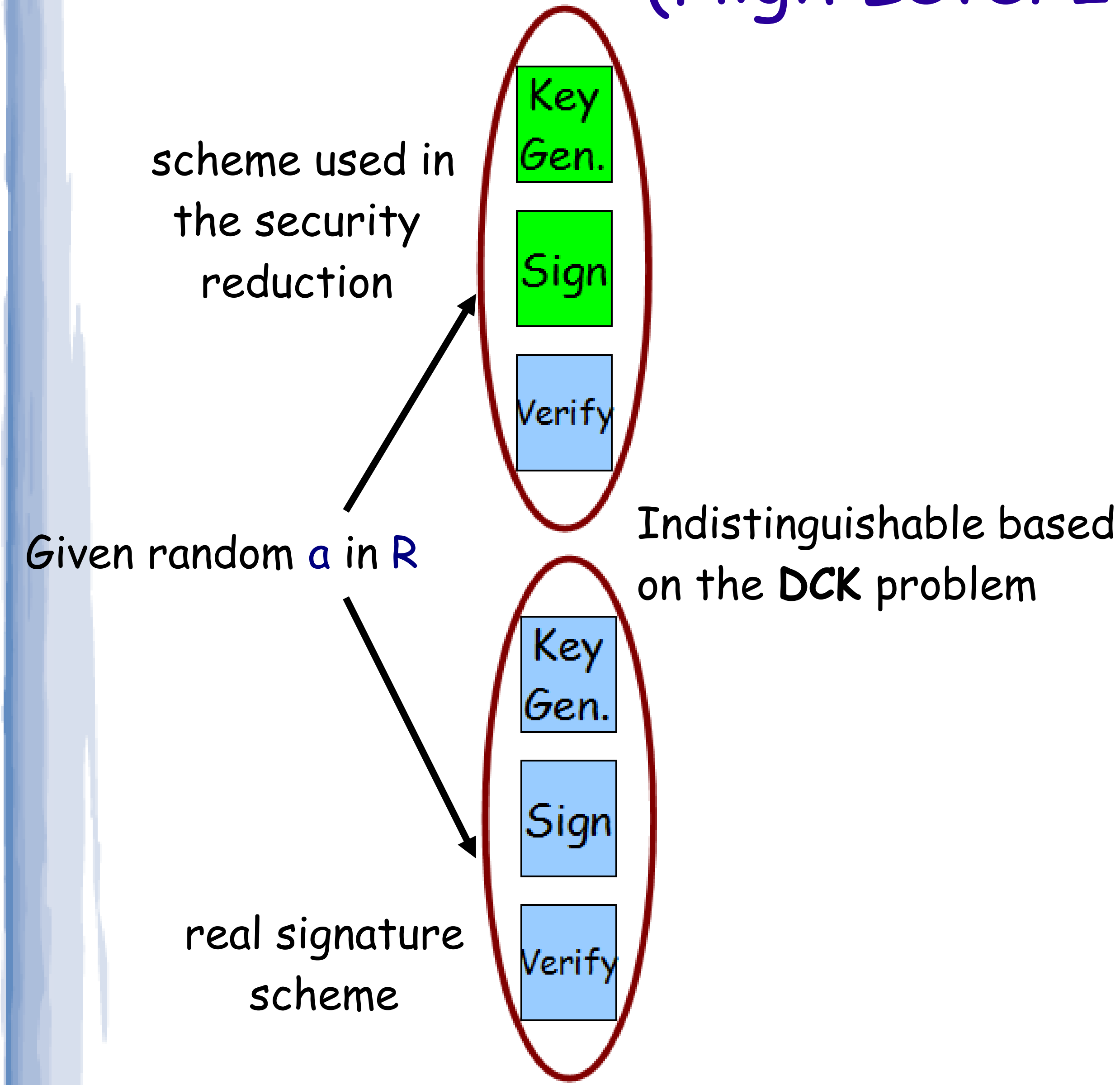

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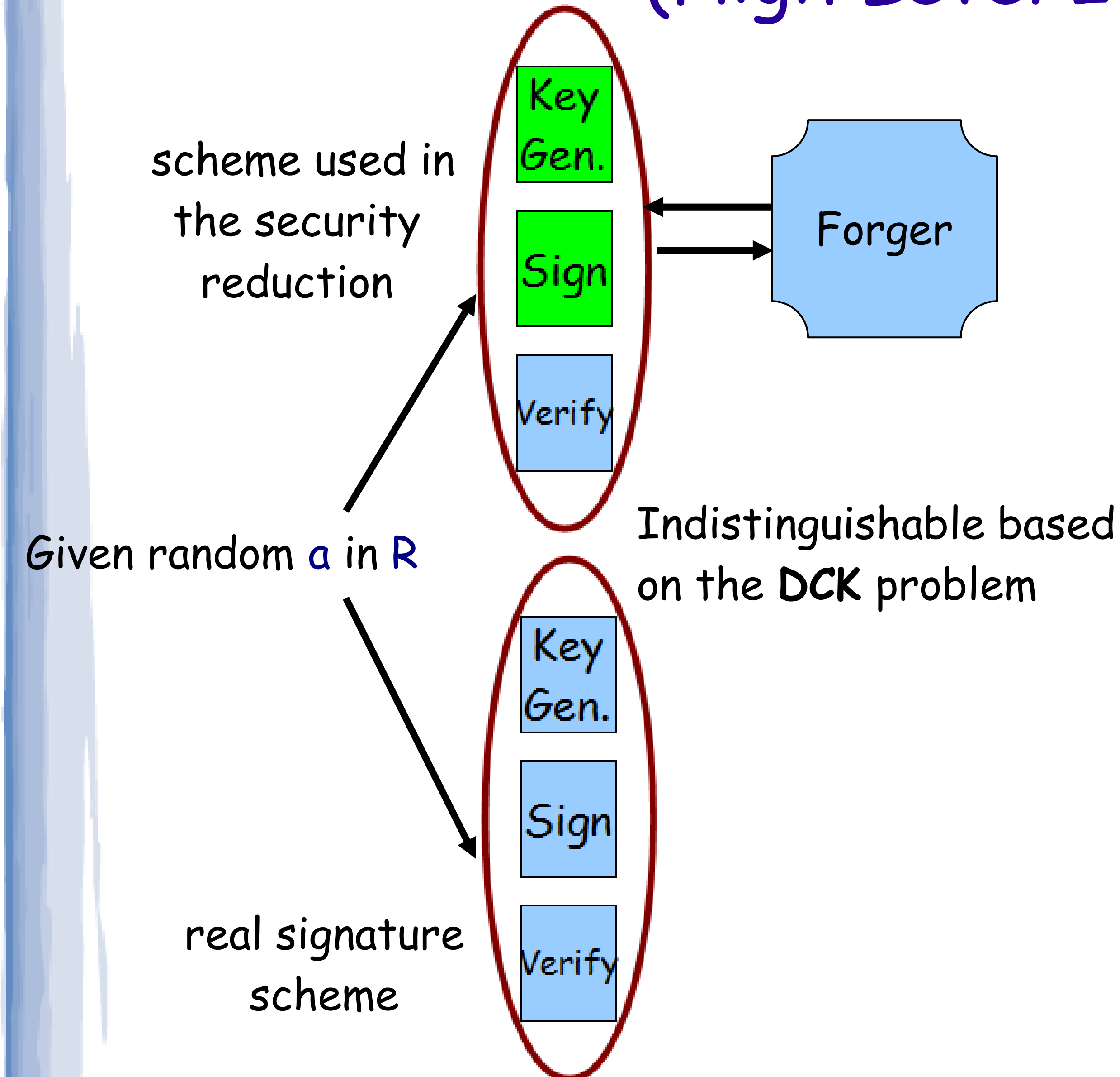

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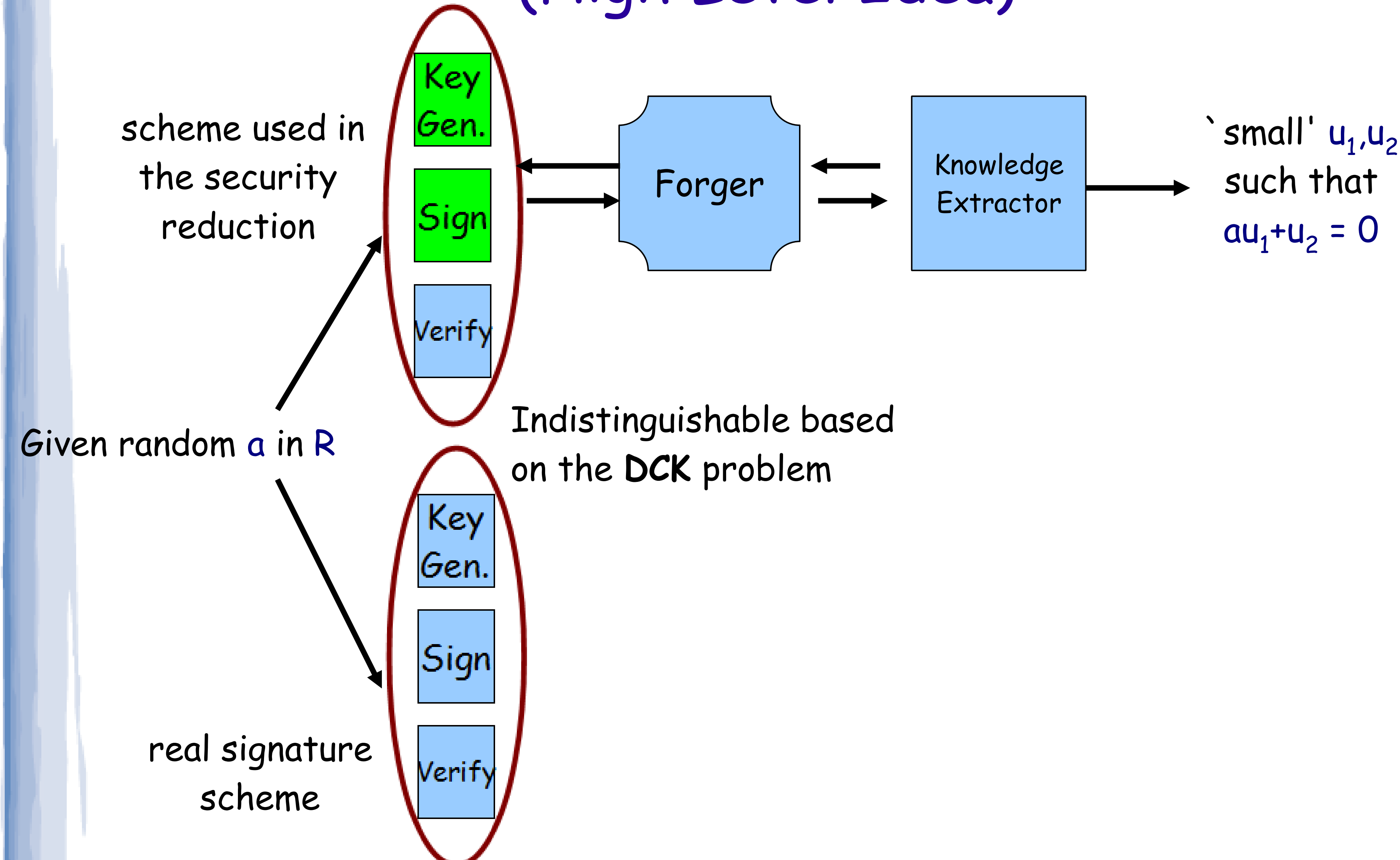

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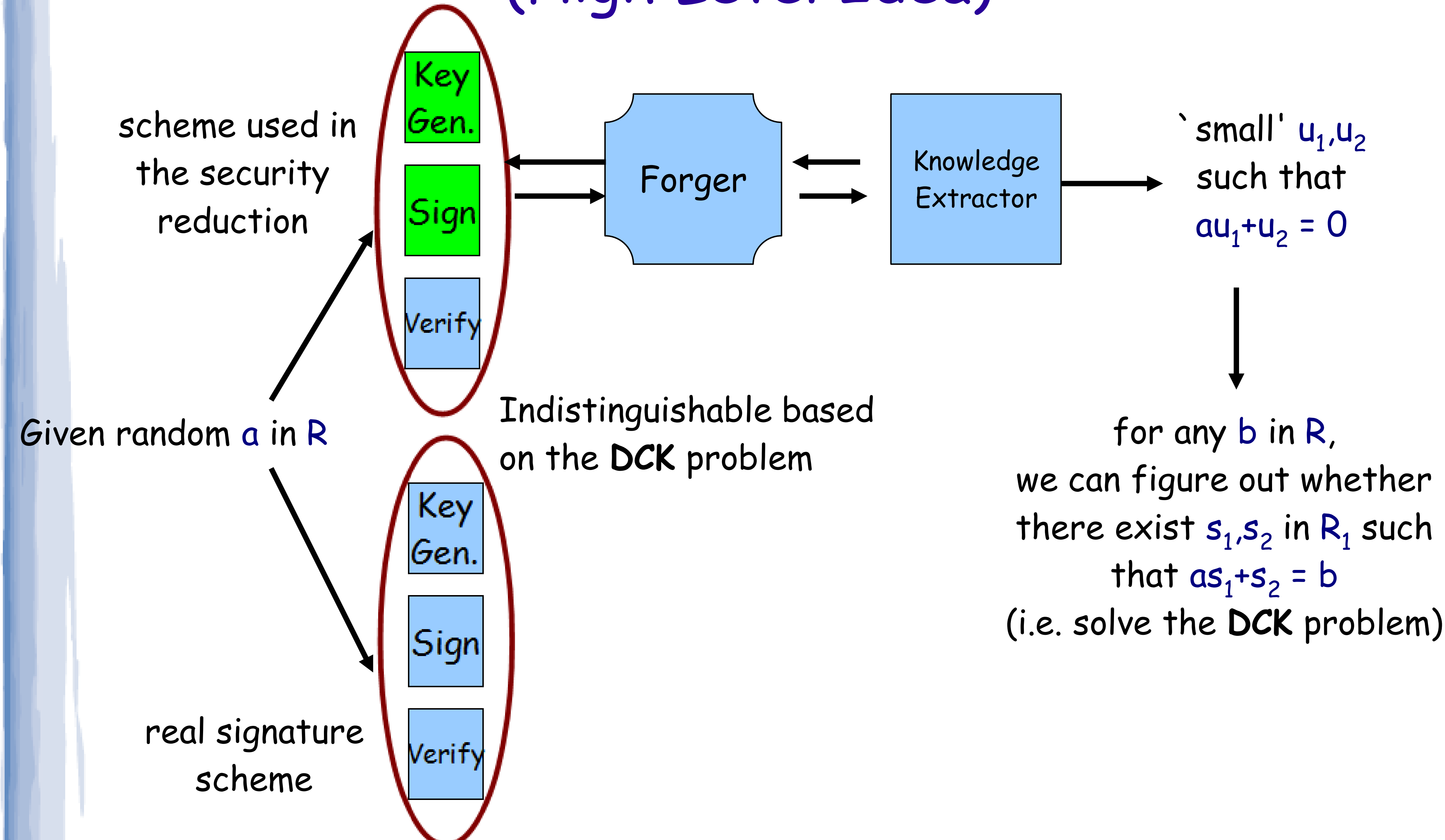

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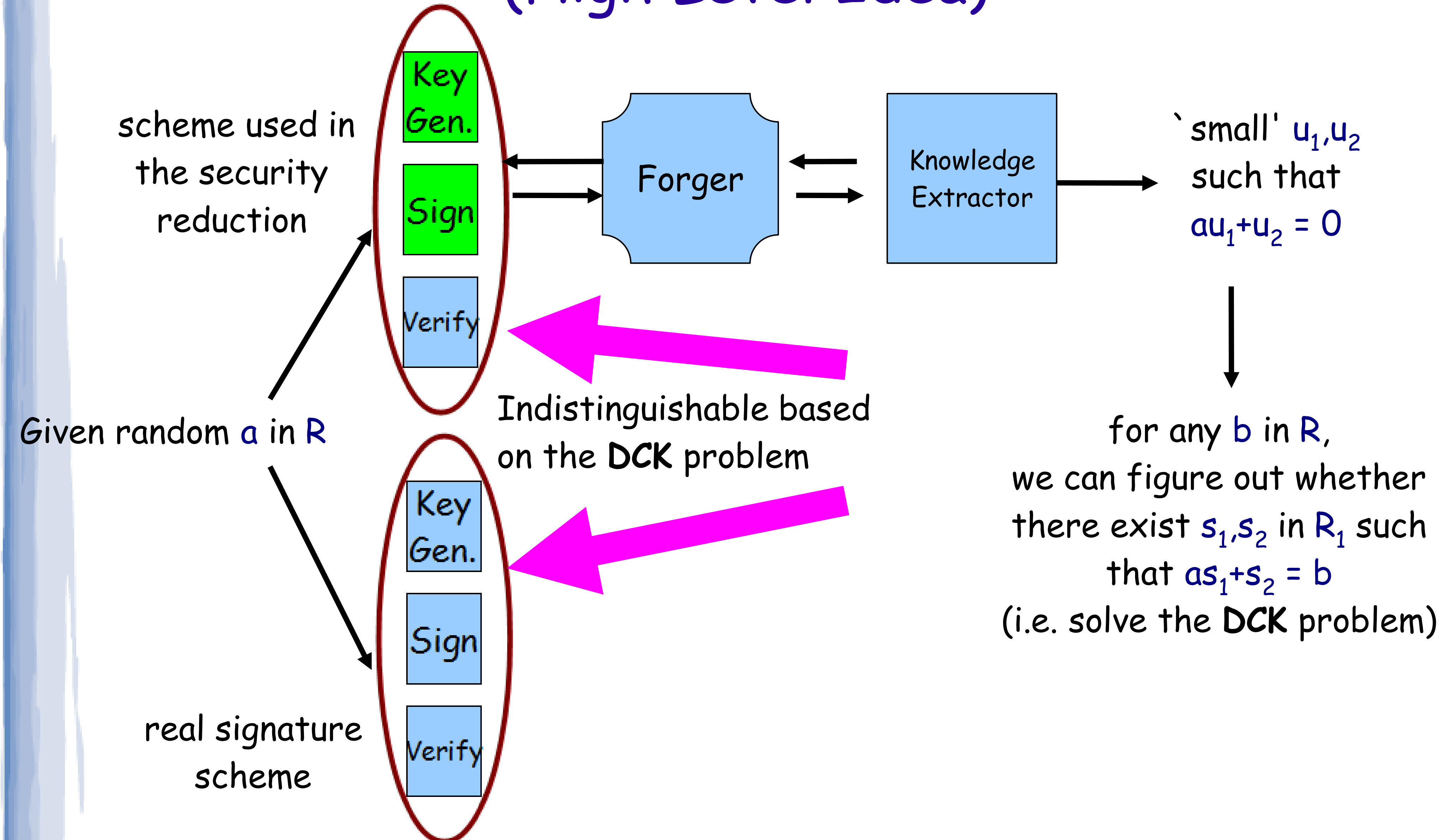

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Pick random z_1,z_2 in R_{k-32}

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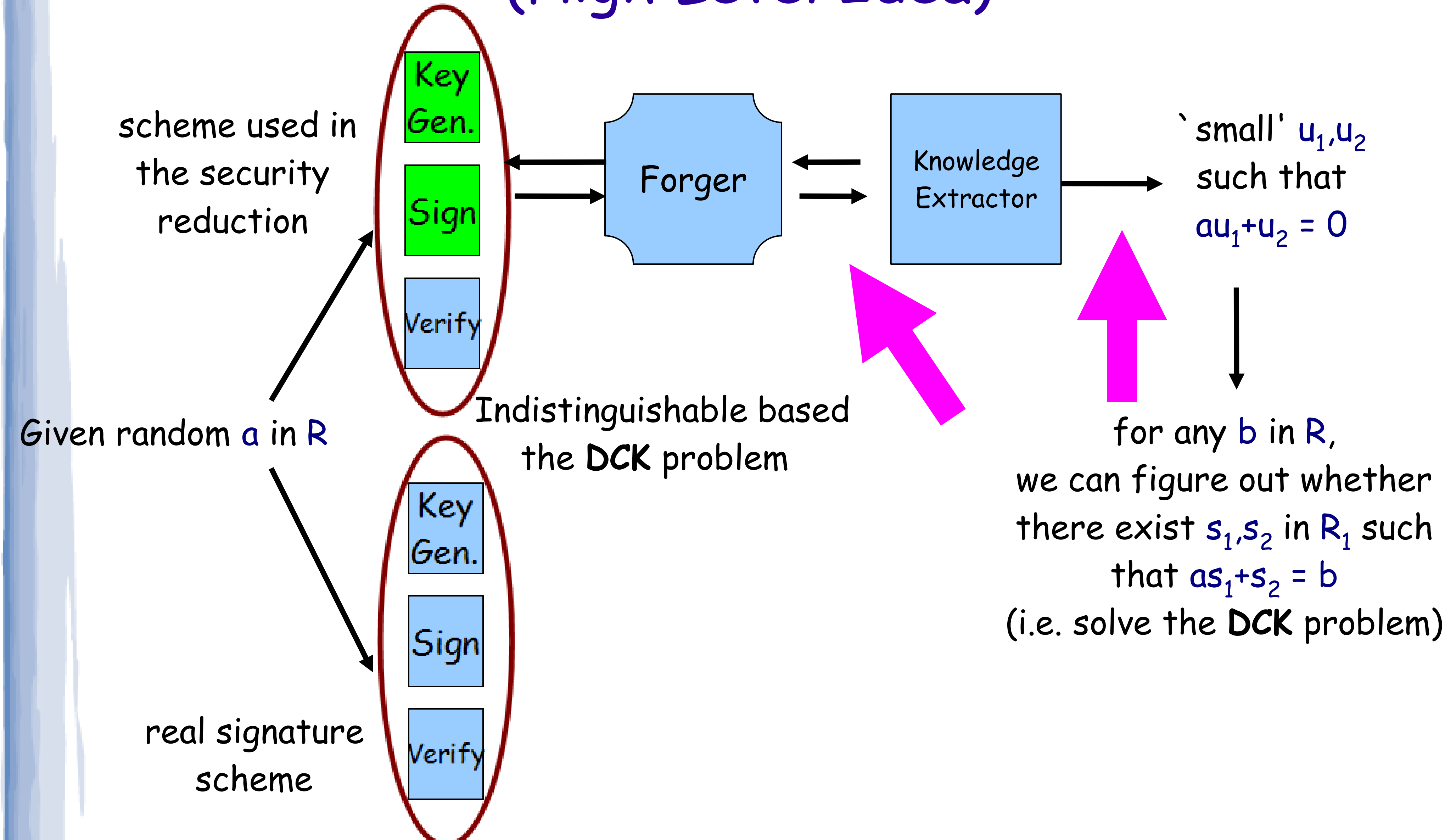
Security Proof
(High Level Idea)



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(z_1,z_2,c) and (z'_1,z'_2,c')

such that

$az_1+z_2 - bc = az'_1+z'_2 - bc'$

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(Because s_1,s_2 are **not** unique, u_1 and u_2 are not both 0)

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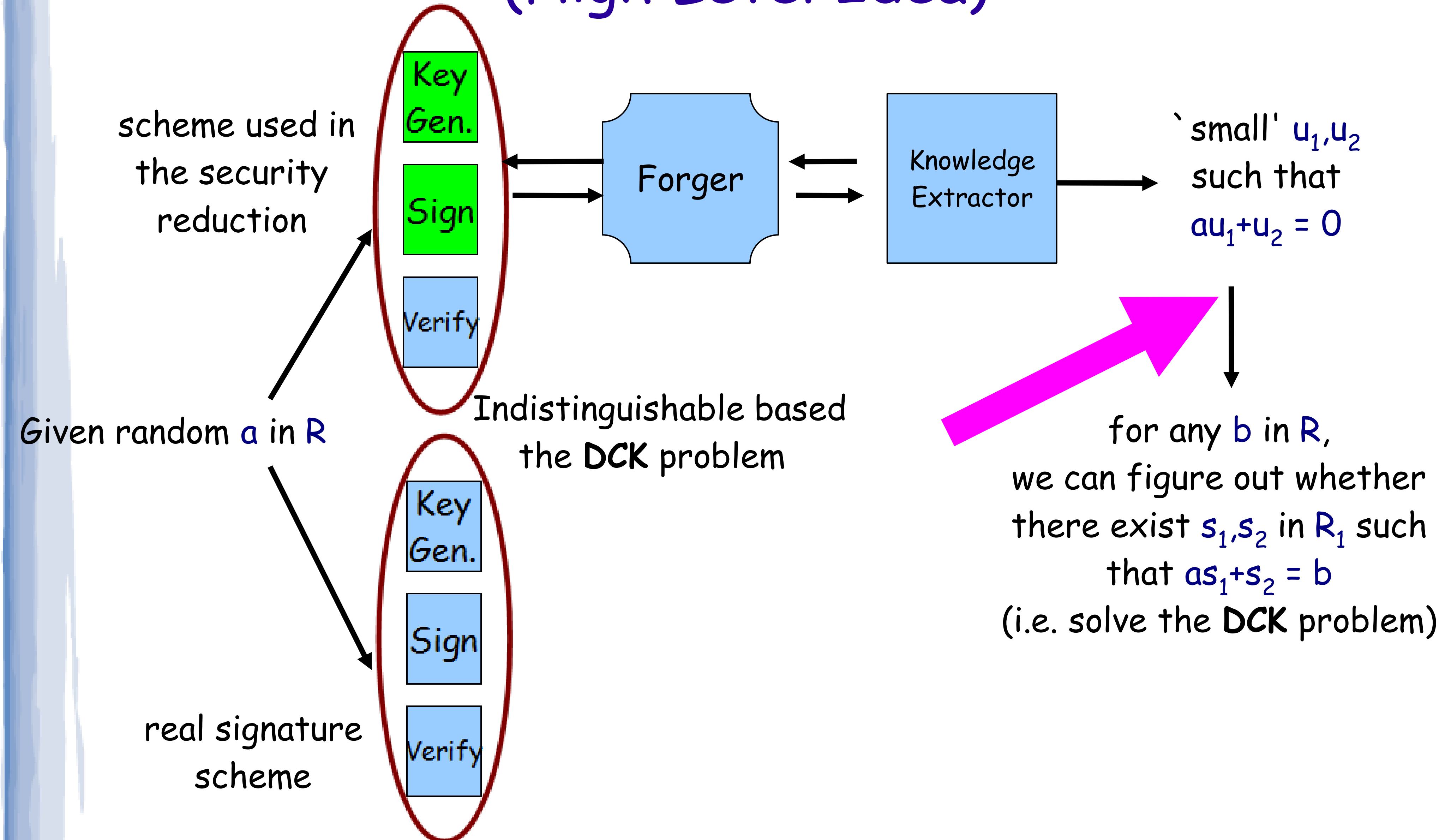

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Given `small' u_1, u_2 such that $au_1+u_2 = 0$, one can solve the DCK problem.

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- If $b = as1 + s2$ for `small' $s1, s2$, then

  $u1b = u1as1 + u1s2 = -u2s1 + u1s2$ is also `small'

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Given `small' u_1, u_2 such that $au_1+u_2 = 0$, one can solve the DCK problem.

Given (a,b), compute u_1b

- If $b={}_a s1_{+s} 2$ for `small' $s1_, s2$, then

  ${}_u1b\ {}_=\ {}_u1as1\ {}_+\ {}_u1s2\ {}_=\ {}_{-u}2s1 + u1s2$ is also `small'

- If b is random, then the coefficients of

  u1b are also random (thus probably `large')

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Thank You!