Trapdoors for Lattices: Signatures, ID-Based Encryption, and Beyond

Chris Peikert
Georgia Institute of Technology

Symbolic Computations and Post-Quantum Cryptography

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Talk Agenda

1. Lattice-based trapdoor functions and preimage sampling
2. Applications: signatures, ID-based encryption (in RO model)
3. ‘Bonsai trees:’ removing the RO & more advanced apps

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3. ‘Bonsai trees:’ removing the RO & more advanced apps

- C. Gentry, C. Peikert, V. Vaikuntanathan (STOC 2008)
  “Trapdoors for Hard Lattices and New Cryptographic Constructions”
- D. Cash, D. Hofheinz, E. Kiltz, C. Peikert (Eurocrypt 2010)
  “Bonsai Trees, or How to Delegate a Lattice Basis”

Main Message

Lattices admit a hierarchy of increasingly powerful 'trapdoors,' which enable many rich applications

Part 1:

Trapdoor Functions and Preimage Sampling

Digital Signatures

(Images courtesy xkcd.org)

Digital Signatures

(public)

(secret)

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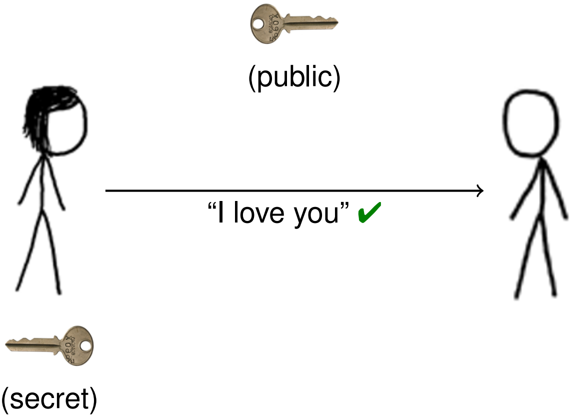
Digital Signatures



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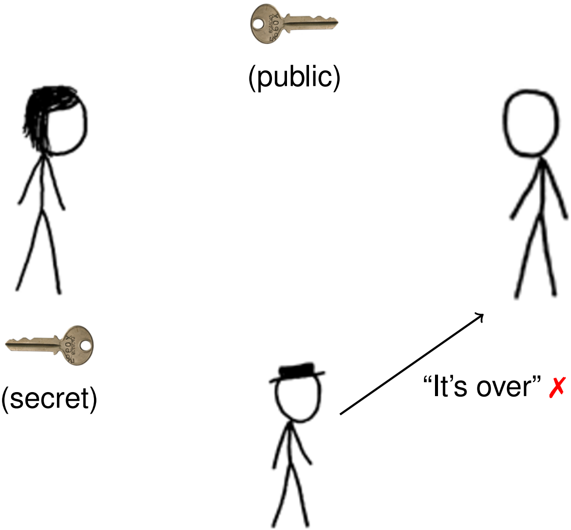

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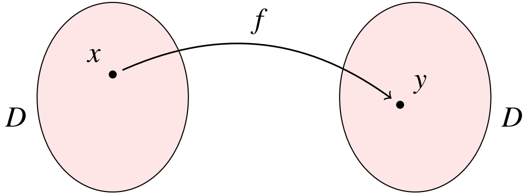

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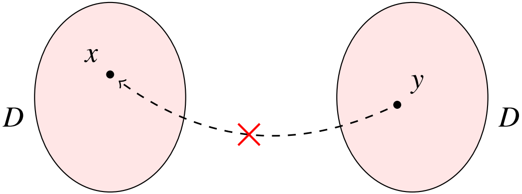

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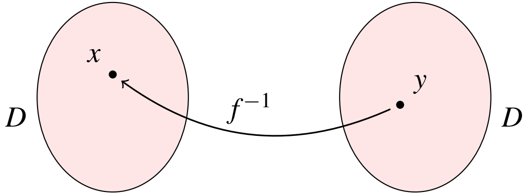

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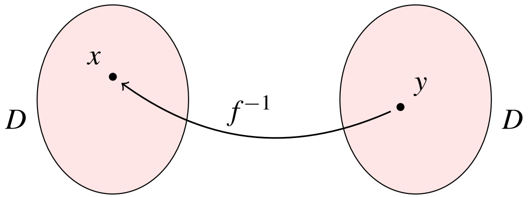

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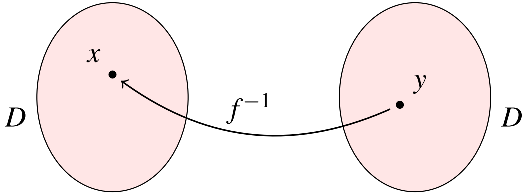

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- Candidate TDPs: [RSA'78,Rabin'79,Paillier'99] ("general assumption")

  All rely on hardness of factoring:

  ✗ Complex: 2048-bit exponentiation

  ✗ Broken by quantum algorithms [Shor'97]

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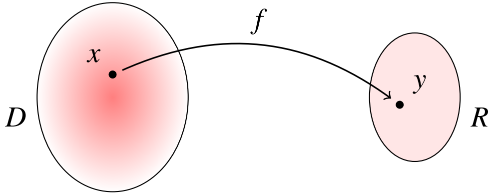

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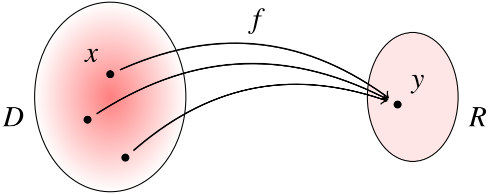

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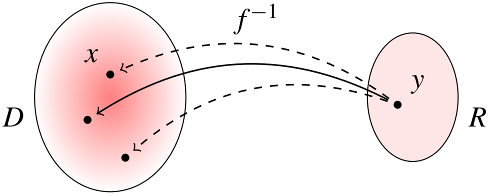

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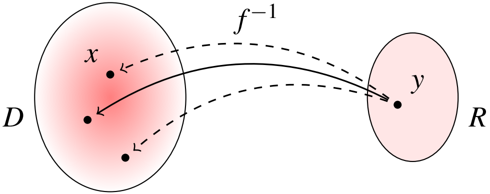

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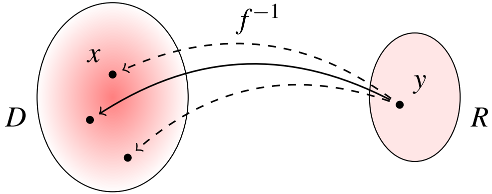
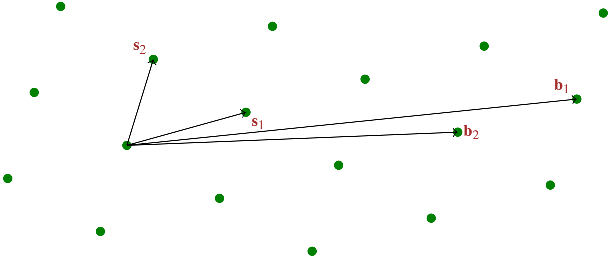


- 'Hash and sign:' $pk = f$, $sk = f^{-1}$. Sign(msg) $= f^{-1}(H(\text{msg}))$.
- Still secure! Can generate (x, y) in two equivalent ways:

<u>REALITY</u>	<u>PROOF</u>
$x \xleftarrow{f^{-1}} y \longleftarrow R$	$D \longrightarrow x \xrightarrow{f} y$

GGH Signatures [GoldreichGoldwasserHalevi'96]

- Key idea: pk = 'bad' basis $\mathbf{B}$ for $\mathcal{L}$, sk = 'short' trapdoor basis $\mathbf{S}$



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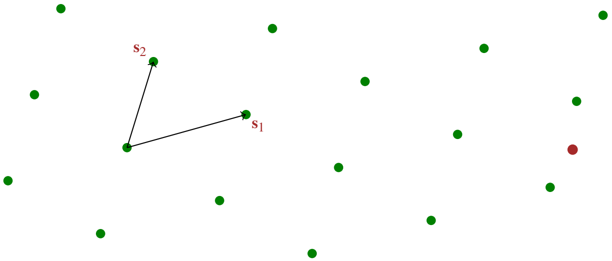

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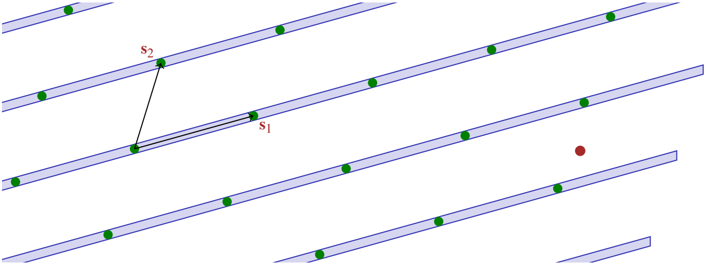

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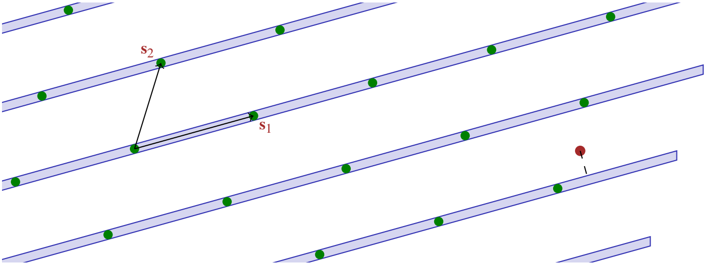

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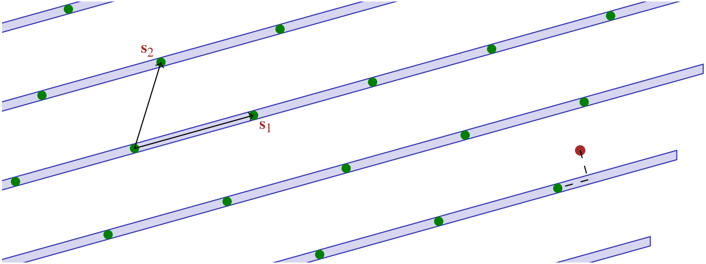

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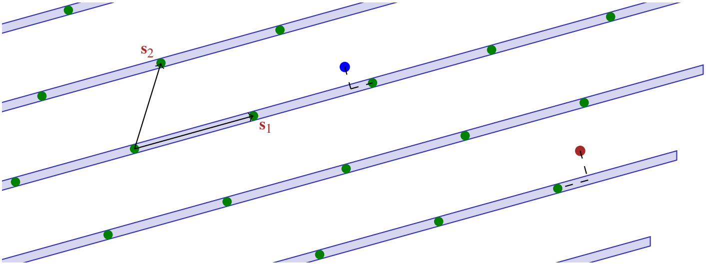

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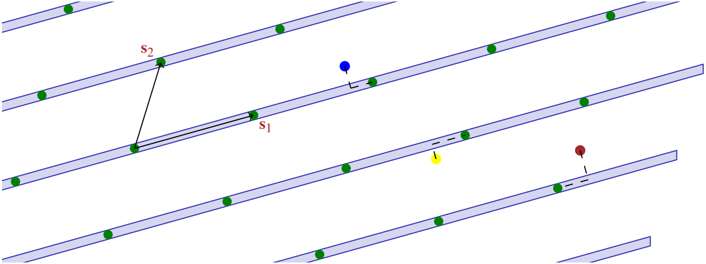

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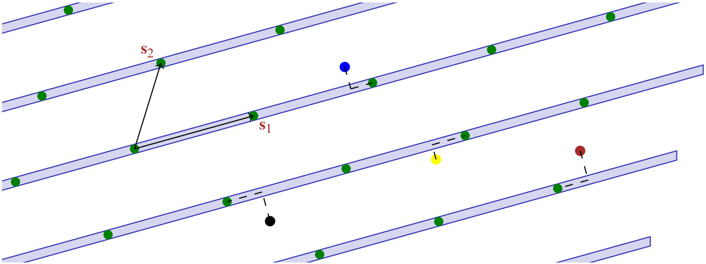

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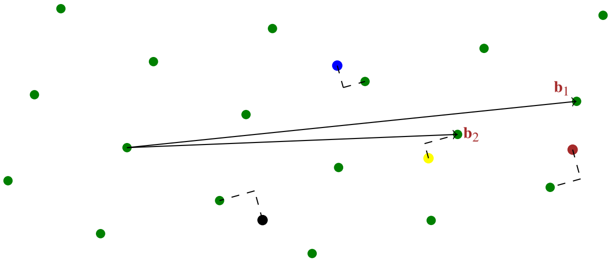

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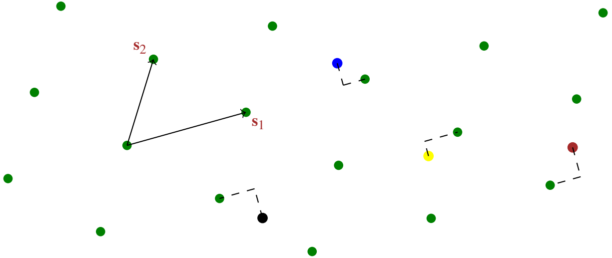

Technical Issues

1. Generating 'hard' lattice together with short basis

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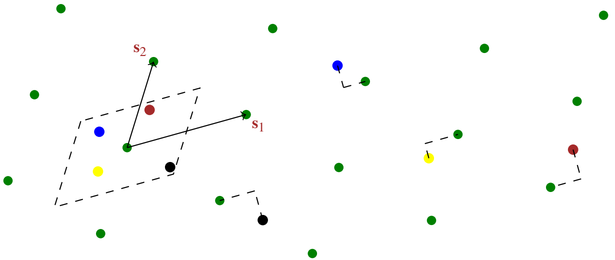

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2. Signing algorithm leaks secret basis!
   - ⋆ Total break after several signatures [NguyenRegev'06]

Blurring a Lattice

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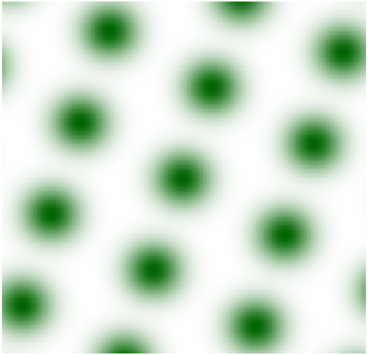

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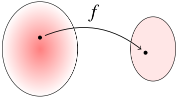
- First used in worst/average-case reductions [Regev’03,MiccReg’04,...]
- Now an essential ingredient in many crypto protocols
  [GPV’08,PV’08,ACPS’09,CHKP’10,OPW’11,...]

Trapdoor Function: Evaluation



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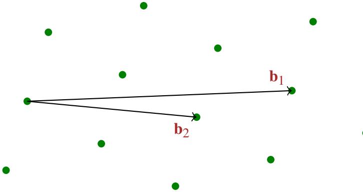

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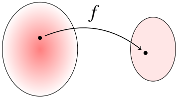

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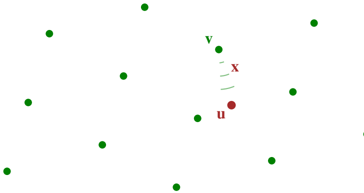

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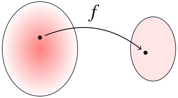

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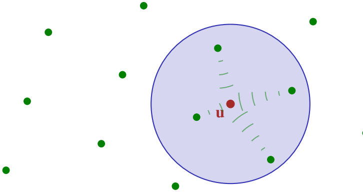

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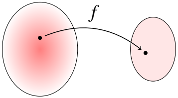

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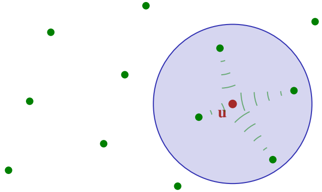

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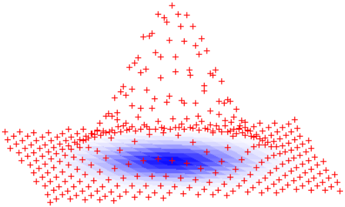

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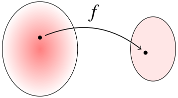

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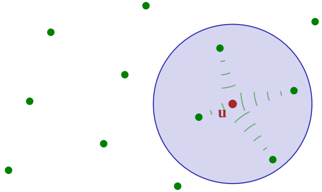

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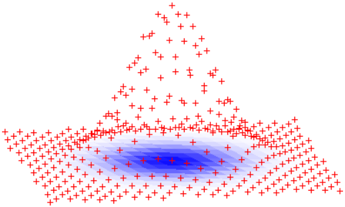

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Typical fact: $\|D_{\mathcal{L},\mathbf{u}}\| \le \sqrt{n} \cdot$ std dev

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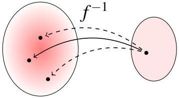

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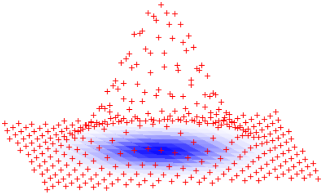

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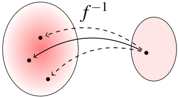

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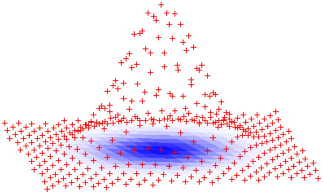

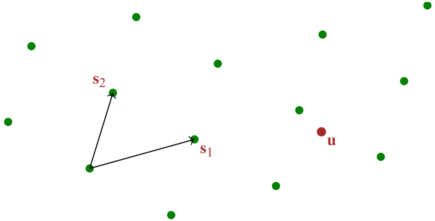

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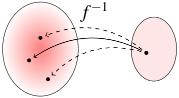

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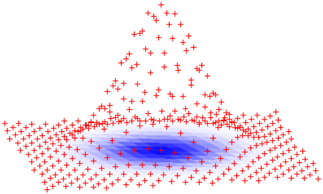

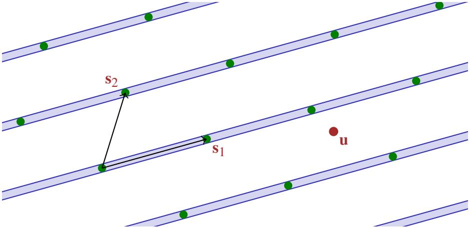

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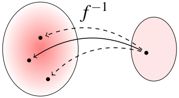

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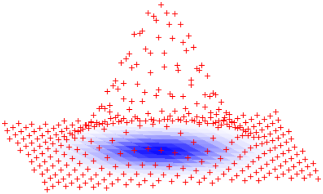

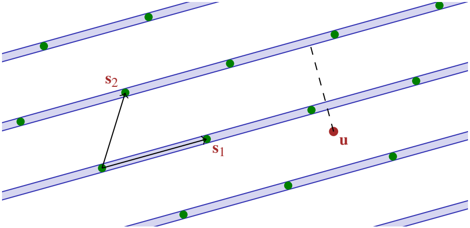

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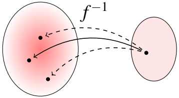

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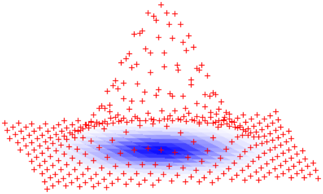

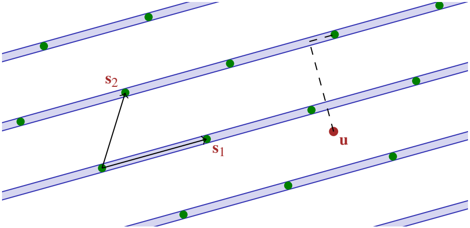

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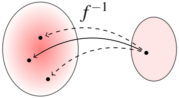

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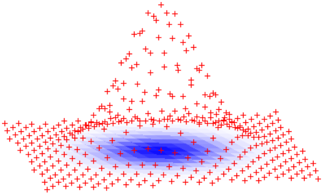

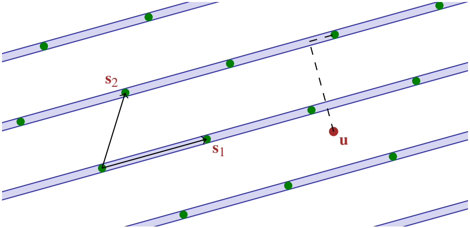

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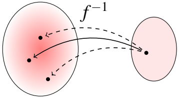

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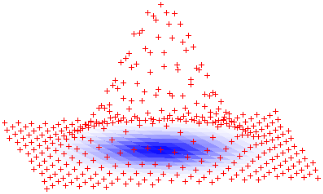

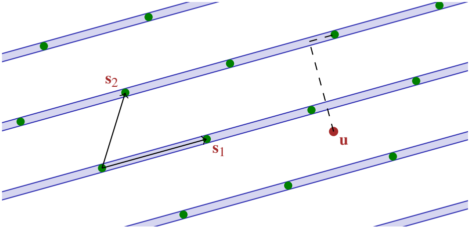

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- [P'10]: Efficient & parallel algorithm for std dev $\geq s_1(\mathbf{S}) \approx \max\|\tilde{\mathbf{s}}_i\|$

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Theorem: Worst-Case/Average-Case [Ajtai'96,...,MR'04,GPV'08]

For uniform $\mathbf{A}$ and $q \geq \beta\sqrt{n}$, finding solution $\mathbf{z} \neq \mathbf{0}$ where $\|\mathbf{z}\| \leq \beta$

$$\Downarrow$$

Solving $\beta\sqrt{n}$-approx GapSVP & more, on any n-dim lattice!

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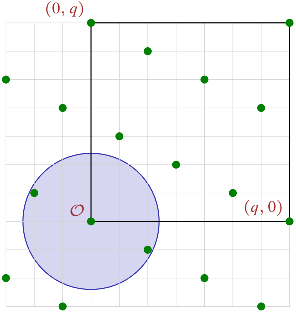

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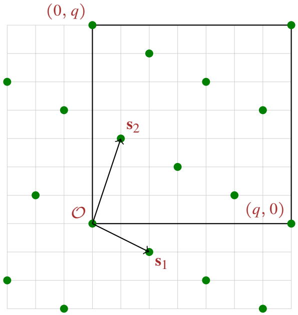

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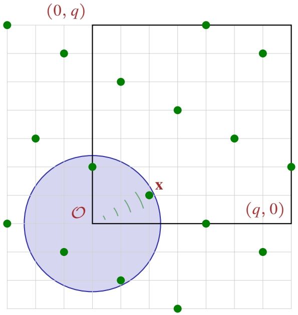

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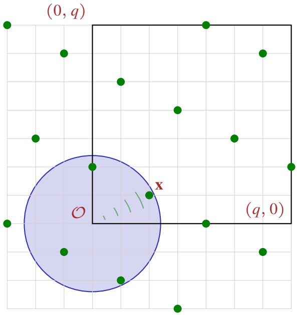

Part 2:

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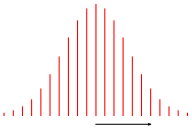

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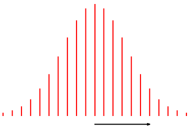

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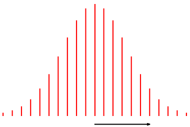

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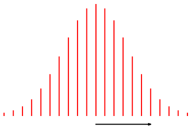

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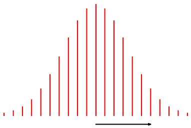

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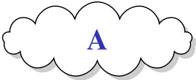
$\Longrightarrow$ $\mathbf{z}$ is a ‘weak’ trapdoor, for distinguishing LWE from uniform

Warm-Up: Public-Key Encryption



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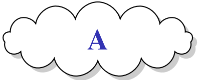

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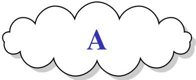

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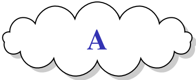

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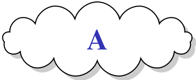

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$\mathbf{s}, \mathbf{e}$ 

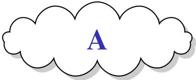
$$\xrightarrow[\text{(public key)}]{\mathbf{u} = \mathbf{A}\mathbf{x} = f_{\mathbf{A}}(\mathbf{x})}$$

$$\xleftarrow[\text{(ciphertext ‘preamble’)}]{\mathbf{b} = \mathbf{A}^t\mathbf{s} + \mathbf{e}}$$

$$\langle \mathbf{x}, \mathbf{b} \rangle \approx \langle \mathbf{u}, \mathbf{s} \rangle \qquad \xleftarrow[\text{(‘payload’)}]{b' + \text{bit} \cdot \lfloor \frac{q}{2} \rfloor} \qquad \boxed{b' = \langle \mathbf{u}, \mathbf{s} \rangle + e'}$$

(‘pad’)

Warm-Up: Public-Key Encryption



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('pad')

- - - - - - - - - - - - - - -

 ? $(\mathbf{A}, \mathbf{u}, \mathbf{b}, b')$

ID-Based Encryption

$\boxed{\mathbf{x} \leftarrow f_{\mathbf{A}}^{-1}(\mathbf{u})}$

$mpk = \mathbf{A}$

$\mathbf{s}, \mathbf{e}$

$\boxed{\mathbf{u} = H(\text{“Alice”})}$

(‘identity’ key)

$$\xleftarrow{\mathbf{b} = \mathbf{A}^t\mathbf{s} + \mathbf{e}}$$

(ciphertext randomness)

$$\langle \mathbf{x}, \mathbf{b} \rangle \approx \langle \mathbf{u}, \mathbf{s} \rangle \qquad \xleftarrow{b' + \text{bit} \cdot \lfloor \frac{q}{2} \rfloor} \qquad b' = \langle \mathbf{u}, \mathbf{s} \rangle + e'$$

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Part 3:

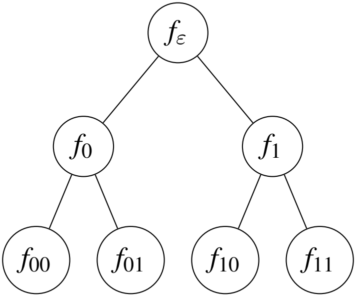
Bonsai Trees:
Removing the Random Oracle and More Advanced Applications

CONTROLLED or NATURAL ?

CONTROLLED or NATURAL ?

- Bonsai: collection of techniques for selective control of tree growth, for the creation of natural aesthetic forms

Bonsai Trees in Cryptography



1. Hierarchy of TDFs

   (Functions specified by public key, random oracle, interaction, . . .)

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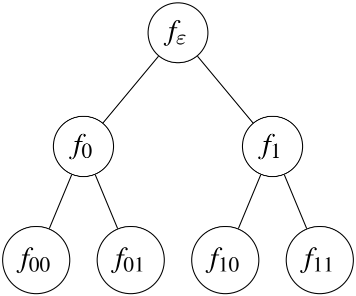

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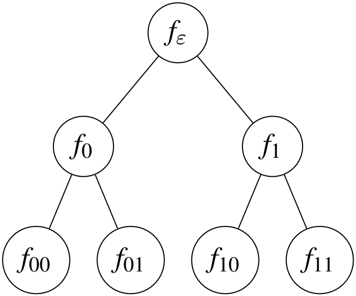

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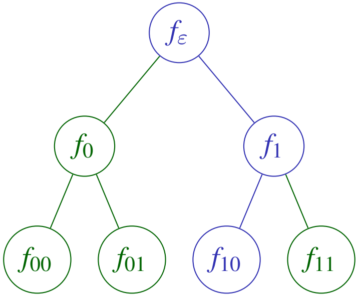
   (Functions specified by public key, random oracle, interaction, . . .)

2. Techniques for selective 'control' of growth & delegation of control

3. Applications: 'hash-and-sign,' (hierarchical) IBE
   — all without random oracles!

Bonsai Trees: Abstract Properties



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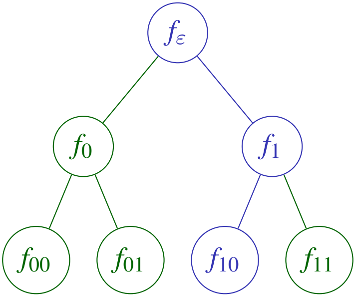

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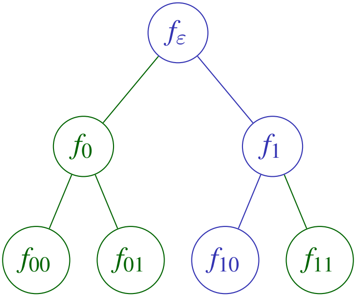

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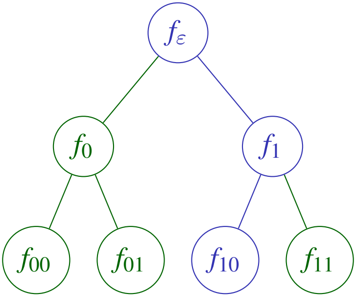

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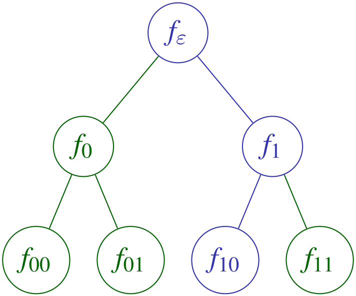

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   (Allows simulation to embed its challenge into the tree, while still being able to answer queries.)
3. Can delegate control of any subtree, w/o endangering ancestors.

Bonsai Trees: Realization

Property 1: Control $f_v \Rightarrow$ Control f_{vz}

Short basis $\mathbf{S}_1$ for $\mathbf{A}_1 \Rightarrow$ short basis $\mathbf{S}$ for $\mathbf{A} = [\mathbf{A}_1 \mid \mathbf{A}_2]$, for any $\mathbf{A}_2$.

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- Just generate $\mathbf{A}_2$ with short basis $\mathbf{S}_2$.

  Then use above technique to control $\mathbf{A}$!

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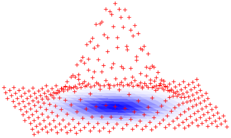
  Solution: Use $\mathbf{S}$ to sample new *Gaussian* basis.



Other Applications of Today's Tools

1. Noninteractive (Statistical) Zero Knowledge [PV'08]
2. Universally Composable Oblivious Transfer [PVW'08]
3. CCA-Secure Encryption [P'09]
4. Many-add, Single-mult Homomorphic Encryption [GHV'10]
5. Bonsai trees with smaller keys [ABB'10]
6. Group signatures [GKV'10]
7. (Bi-)Deniable Encryption [OPW'11]
8. Whatever you can invent!

Closing Thoughts

- A hierarchy of trapdoors for lattices:

  Short vector (decryption)

  $<$ Short basis (sampling)

  $<$ Short basis for 'ancestor' lattice (delegation)

  $< \cdots$

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Thanks!