Multivariate Public Key Cryptography

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Outline

Happy Chinese New Year!

Thank the organizers, in particular, (Alex Myasnikov)^2.

Thank the generous support of the Charles Phelps Taft Memorial Fund and the Taft Family.

Outline

PKC and Quantum computer

- 2519590847565789349402718324004839857142928212620403202777713783604366202070759555626401852588078440691829064124951508218929855914917618450280848912007284499268739280728777673597141834727026189637501497182469116507761337985909570009733045974880842840179742910064245869181719511874612151517265463228221686998754918242243363725908514186546204357679842338718477444792073993423658482382428119816381501067481045166037730605620161967625613384414360383390441495263443219011465754445417842402092461651572335077870774981712577246796292638635637328991215483143816789988504044536402352738195137863656439121201039712282212072035

What is this number?

PKC and Quantum computer

- The number for Microsoft updates

PKC and Quantum computer

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- Digital signature based on RSA

PKC and Quantum computer

- Mathematics behind: integer factorization

$$n = pq.$$

$$15 = 3 \times 5.$$

PKC and Quantum computer

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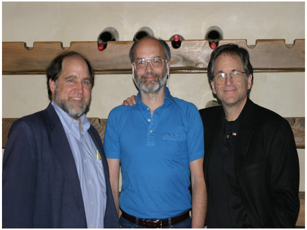
- The concept behind:

Public key Cryptography

PKC and Quantum computer

-  RSA – 2003 Turing prize

Diffie-Hellman – inventors of the idea of PKC

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- Encryption: Public is for encryption and private for decryption
- Digital Signature: Public is for verification and private for signing
- RSA: n is public and p,q is private.
- One knows how to factor n, one can defeat RSA

PKC and Quantum computer

- Quantum computer: using basic particles and quantum mechanics principles to perform computations


R. Feynman

PKC and Quantum computer

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- In 1995, Peter Shor at IBM showed theoretically that it can solve a family of mathematical problems including factoring.

PKC and Quantum computer

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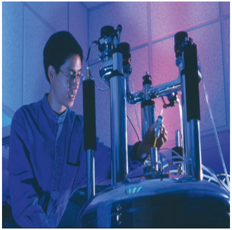
- Isaac Chuang
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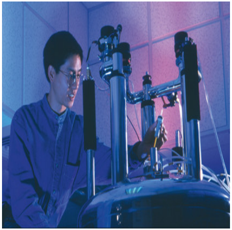
- Isaac Chuang
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- The problem of scaling
  People have different opinions.

PKC and Quantum computer

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PKC and Quantum computer

- PQC – to prepare for the future of quantum computer world
- Ubiquitous computing world .

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  Public key cryptosystems that potentially could resist the future quantum computer attacks. Currently there are 4 main families.

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- 4) Multivariate Public Key Cryptography

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- Multivariate Public Key Cryptosystems
  - *Cryptosystems, whose public keys are a set of multivariate functions*
- The public key is given as:

$$G(x_1, ..., x_n) = (G_1(x_1, ..., x_n), ..., G_m(x_1, ..., x_n)).$$

  Here the G_i are multivariate $(x_1, ..., x_n)$ polynomials over a finite field.

Encryption

- Any plaintext $M = (x_1', ..., x_n')$ has the ciphertext:

$$G(M) = G(x_1', ..., x_n') = (y_1', ..., y_m').$$

Encryption

- Any plaintext $M = (x'_1, ..., x'_n)$ has the ciphertext:

$$G(M) = G(x'_1, ..., x'_n) = (y'_1, ..., y'_m).$$

- To decrypt the ciphertext $(y'_1, ..., y'_n)$, one needs to know a secret (**the secret key**), so that one can invert the map to find the plaintext $(x'_1, ..., x'_n)$.

Toy example

- We use the finite field $k = GF[2]/(x^2 + x + 1)$ with 2^2 elements.

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- We denote the elements of the field by the set $\{\mathit{0}, \mathit{1}, \mathit{2}, \mathit{3}\}$ to simplify the notation.
  Here $\mathit{0}$ represent the 0 in k, $\mathit{1}$ for 1, $\mathit{2}$ for x, and $\mathit{3}$ for $1 + x$.
  In this case, $\mathit{1} + \mathit{3} = \mathit{2}$ and $\mathit{2} * \mathit{3} = \mathit{1}$.

A toy example

- $$\begin{aligned} G_0(x_1, x_2, x_3) &= 1 + x_2 + 2x_0x_2 + 3x_1^2 + 3x_1x_2 + x_2^2 \\ G_1(x_1, x_2, x_3) &= 1 + 3x_0 + 2x_1 + x_2 + x_0^2 + x_0x_1 + 3x_0x_2 + x_1^2 \\ G_2(x_1, x_2, x_3) &= 3x_2 + x_0^2 + 3x_1^2 + x_1x_2 + 3x_2^2 \end{aligned}$$

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- For example, if the plaintext is: $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, then we can plug into G_1, G_2 and G_3 to get the ciphertext $y_0 = 0$, $y_1 = 0$, $y_2 = 1$.

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- This is a bijective map and we can invert it easily. This example is based on the Matsumoto-Imai cryptosystem.

Theoretical Foundation

- Direct attack is to solve the set of equations:

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- Direct attack is to solve the set of equations:

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- *- Solving a set of n randomly chosen equations (nonlinear) with n variables is NP-complete, though this does not necessarily ensure the security of the systems.*

Quadratic Constructions

- *1) Efficiency considerations lead to mainly quadratic constructions.*

$$G_l(x_1, ..x_n) = \sum_{i,j} \alpha_{lij} x_i x_j + \sum_i \beta_{li} x_i + \gamma_l.$$

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- *2) Mathematical structure consideration: Any set of high degree polynomial equations can be reduced to a set of quadratic equations.*

$$x_1 x_2 x_3 = 5,$$

is equivalent to

$$\begin{aligned} x_1 x_2 - y &= 0 \\ y x_3 &= 5. \end{aligned}$$

The view from the history of Mathematics(Diffie in Paris)

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- RSA – Number Theory – the 18th century mathematics
- ECC – Theory of Elliptic Curves – the 19th century mathematics
- Multivariate Public key cryptosystem – Algebraic Geometry – the 20th century mathematics
  Algebraic Geometry – Theory of Polynomial Rings

Early works

- Early attempts by Diffie, Fell, Tsujii, Matsumoto, Imai, Ong, Schnorr, Shamir etc

Early works

- Early attempts by Diffie, Fell, Tsujii, Matsumoto, Imai, Ong, Schnorr, Shamir etc
- Fast development in the late 1990s.

Outline

Multivariate Signature schemes

- **Public key**:
  $G(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$.

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- **Signing a hash of a document**:
  $(x_1, \dots, x_n) \in G^{-1}(y_1, \dots, y_m)$.
- **Verifying**: $(y_1, \dots, y_m) \stackrel{?}{=} G(x_1, \dots, x_n)$.

k, a small finite field.

A toy example over GF(3)

$$
\begin{aligned}
G_1(x_1, x_2, x_3) &= 1 + x_3 + x_1x_2 + x_3^2 \\
G_2(x_1, x_2, x_3) &= 2 + x_1 + 2x_2x_3 + x_2 \\
G_3(x_1, x_2, x_3) &= 1 + x_2 + x_1x_3 + x_1^2
\end{aligned}
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Hash:

$(y_1, y_2, y_3) = (0, 1, 1)$.

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- Direct attack: difficulty of solving a set of nonlinear polynomial equations over a finite field.

How to construct G?

- A scheme by Kipnis, Patarin and Goubin 1999. (Eurocrypt 1999)

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- A scheme by Kipnis, Patarin and Goubin 1999. (Eurocrypt 1999)
- $G = F \circ L$.
  F: nonlinear, easy to compute F^{-1}.
  L: invertible linear, to hide the structure of F.

Unbalanced Oil-vinegar (uov) schemes

- $F = (f_1(x_1, .., x_o, x'_1, ..., x'_v), \cdots, f_o(x_1, .., x_o, x'_1, ..., x'_v))$.

Unbalanced Oil-vinegar (uov) schemes

- $F = (f_1(x_1, .., x_o, x'_1, ..., x'_v), \cdots, f_o(x_1, .., x_o, x'_1, ..., x'_v))$.
- $$f_l(x_1, ., x_o, x'_1, ., x'_v) = \sum a_{lij} x_i x'_j + \sum b_{lij} x'_i x'_j + \sum c_{li} x_i + \sum d_{li} x'_i + e_l$$

  Oil variables: $x_1, ..., x_o$.

  

  Vinegar variables: $x'_1, ..., x'_v$.

How to invert F?

$$f_l(x_1, .., x_o, \underbrace{x'_1, .., x'_v}_{\textbf{fix the values}}) =$$

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- F: linear in Oil variables: $x_1, .., x_o$.

  $\Longrightarrow F$: easy to invert.

Security analysis

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- Direct attacks does not work.

Security analysis

- The mathematical problem to find equivalent secret keys — find the common null subspace spaces of a set of quadratic forms.

$$\begin{array}{cccccc}
0 & .. & 0 & * & .. & * \\
. & . & 0 & * & .. & * \\
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- The problem above can also be transformed into solving a set of quadratic equations.

Rainbow – Ding, Schmidt, Yang etc – ACNS 05,06,07,08

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- Rainbow(18,12,12) over GF(2^8).

$F_1 : o_1 = 12, v_1 = 18.$
$F_2 : o_2 = 12, v_2 = 12 + 18 = 30.$

Rainbow

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Signature 400 bits	**Hash 336 bits**

Implementations

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- FPGA implementation by the research group of Professor Paar at Bochum (CHES 2009)
  Beat ECC in area and speed.

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- **A good candidate for light-weight crypto for small devices like RFID.**

Security

Security

- UOV: not broken since 1999.
- Rainbow – MinRank problem

Pros and Cons

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- Computationally very efficient
- Large public key size

Outline

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- $\bar{K} = k[x]/(g(x))$, a degree n extension of k.
- The standard k-linear invertible map $\phi : \bar{K} \longrightarrow k^n$, and $\phi^{-1} : k^n \longrightarrow \bar{K}$.

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- IP problem.

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- The $\tilde{F}_i = \tilde{F}_i(x_1, \ldots, x_n)$ are quadratic polynomials in n variables. Why quadratic?

$$X^{q^\theta+1} = X^{q^\theta} \times X.$$

Decryption

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- The first toy example is produced by setting $n = 3$ and $\theta = 2$.

Decryption

- The condition: $\gcd(q^\theta + 1, q^n - 1) = 1$, ensures the invertibility of the map for purposes of decryption.
  It requires that k must be of characteristic 2.
- $F^{-1}(X) = X^t$ such that:

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- The first toy example is produced by setting $n = 3$ and $\theta = 2$.
- This scheme is defeated by linearization equation method by Patarin 1995.

HFE by Patarin etc

- The only difference from MI is that F is replaced by a new map given by:

$$F(X) = \sum_{i,j=0}^{D} a_{ij} X^{q^i + q^j} + \sum_{i=0}^{D} b_i X^{q^i} + c.$$

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- D can not too large due to the inversion of using Berlakemp algorithms of solving one variable equations.

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- Construction – small-scale "noise" is added to the system in a controlled way so as to not fundamentally alter the main structure, but yet substantially increase the "entropy."

Internal Perturbation

- Let r be a small integer and

$$z_1(x_1, \ldots, x_n) = \sum_{j=1}^{n} \alpha_{j1} x_j + \beta_1$$

$$\vdots$$

$$z_r(x_1, \ldots, x_n) = \sum_{j=1}^{n} \alpha_{jr} x_j + \beta_r$$

be a set of randomly chosen affine linear functions in the x_i over k^n such that the $z_j - \beta_j$ are linearly independent.

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  be a set of randomly chosen affine linear functions in the x_i over k^n such that the $z_j - \beta_j$ are linearly independent.
- We can use these linear functions to create quadratic "perturbation" in HFE (including MI) systems.

IP of MI

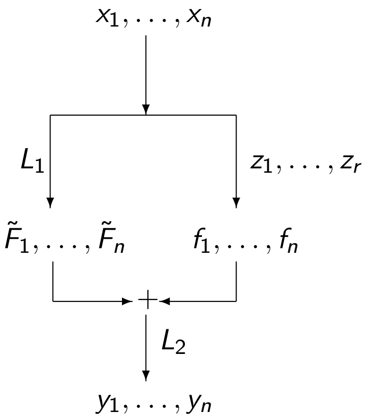

Figure: Structure of Perturbation of the Matsumoto-Imai System.

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- We need to use Plus Method, **Adding random polynomial**, to help it to resist differential attacks.
- Despite the cost of the search, it is still efficient.

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- IPHFE+ (odd characteristics)

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- MFE, TTM

Outline

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- Quantum computer attacks?

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- Some new results are coming out.

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  Heindl, Gao – Diophantine Equations

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  Heindl, Gao – Diophantine Equations
- Other structures

The end

- Thank you very much!
- Questions?