Asymptotic algebra and modern cryptanalysis

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Search for generically hard problems

Plan:

- Generic Properties of van Kampen diagrams
- Asymptotically dominant subgroups
- Cryptanalysis of group based cryptosystems

Generic properties of van Kampen diagrams.

Let X be a finite set (generators), R a finite set of words in the alphabet $(X \cup X^{-1})^*$ (relators),

$P = \langle X; R \rangle$ a finite presentation (of a group G).

$\mathcal{D}(P)$ - the set of all van Kampen diagrams over P equipped with the measure constructed above.

Our goal: to study generic properties of van Kampen diagrams over P.

A property $\mathcal{C}$ is **generic** if the set $\mathcal{D}_{\mathcal{C}}$ of van Kampen diagrams satisfying $\mathcal{C}$ is generic in $\mathcal{D}(P)$.

Linear isoperimetric inequalities and hyperbolicity

A van Kampen diagram D over P satisfies the **linear isoperimetric inequality** with coefficient c if

$$Area(D) \leq cL(D)$$

where $area(D)$ is the area of D and $L(D)$ is the length of the boundary (perimeter) of D.

Gromov: A group G given by a finite presentation P is hyperbolic if and only if there exists a constant c such that every reduced van Kampen diagram over P satisfies the linear isoperimetric inequality with the coefficient c.

Generic hyperbolicity

Theorem [M.-Ushakov] The set of all van Kampen diagrams satisfying the linear isoperimetric inequality with coefficient 4

$$area(D) \leq 4L(D)$$

is strongly generic in $\mathcal{D}(P)$.

Global and local generic hyperbolicity

Generic Global Hyperbolicity [Gromov]: A generic finitely presented group is hyperbolic.

Generic Local Hyperbolicity [M.-Ushakov]: Every finitely presented group is generically hyperbolic.

Generic depth of van Kampen diagrams.

Theorem [M.-Ushakov] The set of all van Kampen diagrams satisfying the logarithmic depth-area inequality

$$Depth(D) \leq \log Area(D)$$

is generic in $\mathcal{D}(P)$.

Corollary. The set of all van Kampen diagrams satisfying the logarithmic depth-perimeter inequality

$$Depth(D) \leq \log L(D) + \log 4$$

is generic in $\mathcal{D}(P)$.

Generic complexity of the word problem

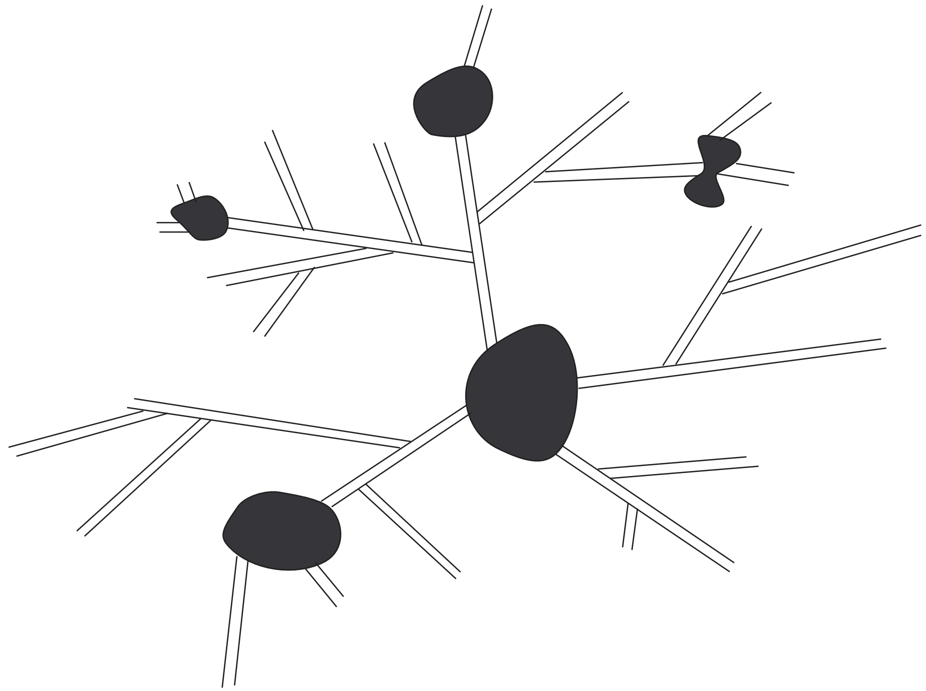
Theorem [M.-Ushakov] For a given finitely presented group $G = \langle X; R\rangle$ the algorithm $\mathcal{A}_W$ solves the Search Word Problem in G generically in polynomial time.

Indeed, recall the complexity of the algorithm $\mathcal{A}_W$:

$$O(|w|L(R)^{Depth(w)})$$

where $L(R)$ is the total length of the presentation of G, and $Depth(w)$ is the minimal depth of van Kampen diagrams presenting w.

Structure of a random van Kampen diagram.



Asymptotically dominant subgroups and Cryptanalysis

Anshel-Anshel-Goldfeld scheme

Public: Group G with two f.g. subgroups

$A = \langle a_1, \ldots, a_m \rangle, \quad B = \langle b_1, \ldots, b_n \rangle$

Alice: takes a (secret) word $a = u(a_1, \ldots, a_m)$ in alphabet $A^{\pm 1}$, encodes (by normal forms), and makes public:

$$b_1^a, \ldots, b_n^a$$

Bob: takes a (secret) word $b = v(b_1, \ldots, b_n)$ in alphabet $B^{\pm 1}$, encodes (by normal forms), and makes public:

$$a_1^b, \ldots, a_m^b.$$

Shared **secret key**: $\mathbf{a^{-1}a^b} = [a, b] = \mathbf{(b^a)^{-1}b}$

Attacks on AAG schemes

- Conjugacy problem
- Normal forms
- Subgroups

Search for the Platform G

Known platforms:

- Braid groups,
- Thompson groups,
- Miller groups,
- Polycyclic groups,
- Free metabelian groups.

Naive attempts

Attempt 1:

$G = F(A)$ - free group on $A = \{a_1, \ldots, a_n\}$.

The Conjugacy Problem in free groups

$u^x = v$ in $F(A)$:

Algorithm:

- find $\tilde{u}$ of minimal length among all conjugates of u,
- find $\tilde{v}$ of minimal length among all conjugates of v,
- $u^x = v \iff \tilde{v}$ is a cyclic permutation of $\tilde{u}$ and x is an initial segment of $\tilde{u}$.

Conjugacy Game (length based attack):

Player 1: for a given u has to find $\tilde{u}$. The word u is not known to him but he may ask questions about the length of elements.

Player 2: does the required manipulations and tells the length of the resulting words

Winning strategy for Player 1:

- find $b_1 \in A^{\pm 1}$ such that $l(u^{b_1}) < l(u)$
- repeat for $u_1 = u^{b_1}$
- output $x = b_1 \dots b_k$

Whitehead-type attacks on CP

$u^x = v$ in a group G:

- find "minimal length" elements $\tilde{u}$, $\tilde{v}$ in the conjugacy classes of u and v
- solve CP for $\tilde{u}$, $\tilde{v}$ (typically the conjugator is short)

Naive Search for the Platform

Attempt 2: $G = F(A)$ - free group on A as before.

But G is given by a finite non-standard presentation:

$$G = \langle X; R \rangle$$

The Length-Based Attack will work if we can compute the length of elements in G relative to the standard presentation $G = \langle A; \emptyset \rangle$

Geodesic length of elements

Let $G = \langle X; R \rangle$

w is an element of G given as a word in $X \cup X^{-1}$

The geodesic length $L(w)$ of w is the length of a minimal path from 1 to w in the Cayley graph of G (with respect to the generating set X).

$$L(w) = \text{min}\{|u| \mid u =_G w\}$$

The Word Problem is decidable in $G \iff$ the function $L : w \to L(w)$ is computable.

The geodesic length problem

Geodesic Length Problem: Given $G = \langle X; R \rangle$ and a word w in $X \cup X^{-1}$ compute the geodesic length $l(w)$ of the element w.

If G has undecidable word problem?

Compute the length on a generic subset.

Moreover, compute the length approximately.

Cryptanalysis of Attempt 2

Attempt 2: $G = F(A)$ is a free group on A given by a finite non-standard presentation:

$$G = \langle X; R \rangle$$

The length-Based Attack works if one can "compute" the geodesic length function $L(w)$.

Known Platforms: Braid Groups

B_n - the group of n-string braids.

Classical Artin presentation:

$$B_n = \left\langle \, x_1, \ldots, x_{n-1} \;\middle|\; \begin{array}{ll} x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, & 1 \le i \le n-2 \\ x_i x_j = x_i x_j, & |i-j| > 1, \;\; 1 \le i, j \le n-1 \end{array} \right\rangle$$

Normal Forms in Braid Groups

Garside normal forms:

Time complexity to compute the normal form of $b \in B_n$ is bounded by $O(|b|^2 n \log n)$.

Dehornoy normal forms:

Time complexity: not known (believed to be "fast" on most inputs)

Conjugacy Problem

- CP is decidable, but the time complexity is not known
- Not known to be of polynomial time.
- Known to be of polynomial time on "most" inputs

Subgroup attacks

Main ideas:

- "Random" subgroups of a given group are "very particular"
- CP is very particular in random subgroups
- Length base attacks

Random Subgroups

Let G be a group with a finite set of generators X.

Main Question: What are random subgroups in G?

No subgroup Cayley Graphs, no random walks on them

Random Generator of subgroups in G:

- pick a random $k \in \mathbb{N}$

- pick randomly k words $w_1, \ldots, w_k \in F(X)$

- generate a subgroup $\langle w_1, \ldots, w_k \rangle$ of G.

Asymptotically visible subgroups

Fix $k \in \mathbb{N}$. For $t \in \mathbb{N}$ put

$$Sub_t(X,k) = \{(w_1,\dots,w_k) \mid w_i \in F(X), |w_i| \le t\}$$

For a group H put

$$Sub_t(G,X,H,k) = \{(w_1,\dots,w_k) \in Sub_t(X,k) \mid \langle w_1,\dots,w_k\rangle \simeq H\}$$

Then the ratio

$$f_t(G,X,H,k) = \frac{\mid Sub_t(G,X,H,k) \mid}{\mid Sub_t(X,k) \mid}$$

gives the frequency of subgroups isomorphic to H that occur among all subgroups generated by k-tuples of words in $X^{\pm 1}$ of length at most t.

H is **asymptotically dominant in** G if

$$\limsup_{t\to\infty} f_t(G, X, H, k) \neq 0.$$

k-**spectrum** of G:

$$Spec_k(G) = \{dominant\ subgroups\ of\ G\ \}$$

Main problem: What is $Spec_k(G)$ for a given G?.

Generic subgroups

For $k \in \mathbb{N}$ we say that a k-generated group H is **a generic subgroup** of G if a random k-generated subgroup of G is isomorphic to H:

$$\limsup_{t \to \infty} f_t(G, X, H, k) = 1$$

In this case we say that G has **unique random k-subgroups** (URS_k).

Main Themes:

1. For a given group G and $k \in \mathbb{N}$ describe the spectrum $Spec_k(G)$;

2. Study groups with URS.

Asymptotically Dominant Nielsen Property

A group G generated by X has **asymptotically dominant Nielsen property** if for every $k \in \mathbb{N}$ a random tuple $W = (w_1, \ldots, w_k) \in F(X)$ generates a free group and W is a Nielsen of the group.

In groups with Dominant Nielsen property **free groups are generic** subgroups.

Examples of groups with DNP:

- Free groups [Jitsukawa]

- Pure Braid groups [Myasnikov, Ushakov]

- Hyperbolic groups [Gilman, Myasnikov, Osin]

- Groups admitting a non-trivial splitting as an amalgamated free product or HNN extension

Length Based Attacks in groups with DNP

In AAG scheme one chooses random subgroups:

$$A = \langle a_1, \ldots, a_n \rangle, \quad B = \langle b_1, \ldots, b_m \rangle$$

Then solve the conjugacy equations of the type $w^x = w^*$ in the group generated by $C = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ which is free on C.

Length base attack!

But we need to know the length...

Computing the length

G generated by X.

Geodesic normal forms in G:

$$w \to \bar{w}$$

if $\bar{w}$ is a shortest representative of w (geodesic in the Cayley graph of G relative to X).

Computing of geodesic normal forms in G gives the length function d (distance) in G.

Sometimes normal forms are not geodesic but **quasi-geodesic**.

This gives an approximation d^* of the length function on G.

Approximating the length in Braid Groups

[Dynnikov]

Asymptotically Dehornoy forms give some approximation d^* of the length in braid groups B_n.

[Myasnikov, Shpilrain, Ushakov]

"Practical" approximation of the length in B_n based on Dehornoy forms and heuristic algorithms.

Length functions in subgroups

H is a subgroup of G generated by W

The length d_W in H is **different** from the length d_X induced from G.

How to realize the length attack on H even if H is free?

Possible if H is **quasi-isometrically** embedded in G: d_X is an approximation of d_W.

Quasi-isometrically embedded subgroups

Conjecture:

Property of being quasi-isometrically embedded is asymptotically dominant in groups.

In hyperbolic groups generic subgroups are free and quasi-isometrically embedded.

Large subgroups

Idea: instead of long generators use "short" generators of subgroups in G.

For a fixed $t \in \mathbb{N}$ we say that t-short subgroups **converge to** G if "most" of the t-short subgroups are equal to G.

There are very effective attacks if the subgroup are equal to G.

Subgroup Black Holes

Idea: Chose subgroups in the subgroup black hole

Good subgroups in Braid groups