On the strongly generic undecidability of the halting problem

Alexander N. Rybalov
Omsk State University
rybalov@omskreg.ru

November 21, 2005

The halting problem (HP)

- Input: A Turing machine M
- Output:

  YES if M halts on $\delta(M)$

  NO if M does not halt on $\delta(M)$

here δ is some effective coding of Turing machines by binary strings

Question 1 *Is there an algorithm deciding HP?*

Theorem 1 (Classics) *HP is algorithmically undecidable.*

Generic-case version of the HP

- Input: "Almost every" Turing machine M

- Output:

  YES if M halts on $\delta(M)$

  NO if M does not halt on $\delta(M)$

Question 2 *Is there an algorithm deciding HP for "almost all" inputs?*

Question 3 *What does it mean "almost all"?*

Asymptotic density of sets of programs

- P is the set of all Turing machines
- P_n is the set of all n-state machines
- B is some set of Turing machines

Definition 1 *Asymptotic density of B is*

$$\mu(B) = \overline{\lim_{n\to\infty}} \frac{|B \cap P_n|}{|P_n|}.$$

The number of all n-state programs

Working alphabet is $\Sigma = \{0, 1, \Box\}$. Machine can move the head to left and to right cell of the tape. Every n-state program contains $3n$ rules of type

$$(q_i, a) \to (q_j, b, s),$$

for every state q_i, $i = 1, \ldots, n$ and every symbol $a \in \Sigma$. Here $a, b \in \Sigma$, $s \in \{L, R\}$ and q_j may be final state.

This follows that the number of all n-state programs is

$$|P_n| = (6(n + 1))^{3n}.$$

Generic sets of programs

Definition 2 *A set B of programs is called*

- *generic if $\mu(B) = 1$*

- *negligible if $\mu(B) = 0$*

- *strongly negligible if there are constants $0 < \sigma < 1$ and $C > 0$ such that for every n*
  $$\frac{|B \cap P_n|}{|P_n|} < C\sigma^n,$$
  i.e. the sequence of the proportion of all n-state programs in B exponentially fast converges to 0

- *strongly generic if $P \setminus B$ is strongly negligible*

Generic-case decidability and complexity of HP

Question 4 *Is there a generic set of Turing machines on which the HP is decidable?*

Theorem 2 (Hamkins, Miasnikov) *There is a generic set of Turing machines B such that HP is polynomial time decidable on B.*

Question 5 *What about strongly generic sets on which HP is decidable?*

Theorem 3 (Main result) *There is no strongly generic set of Turing machines on which HP is decidable.*

How do we prove undecidability of classical HP?

Suppose HP is decidable, then

$$halt(x) = \begin{cases} 1, & \text{if } x = \delta(M) \text{ and } M(x)\downarrow, \\ 0, & \text{if } x = \delta(M) \text{ and } M(x)\uparrow. \end{cases}$$

is computable function on $\delta(P)$. Then the "diagonal" function

$$diag(x) = \begin{cases} \text{not def}, & \text{if } x = \delta(M) \text{ and } M(x)\downarrow, \\ 0, & \text{if } x = \delta(M) \text{ and } M(x)\uparrow. \end{cases}$$

is computable on $\delta(P)$ too. But the machine M computing $diag$ makes an error on $\delta(M)$:

if $M(\delta(M))\downarrow \Rightarrow diag(\delta(M)) = 0 \Rightarrow M(\delta(M))\uparrow$.

if $M(\delta(M))\uparrow \Rightarrow diag(\delta(M))$ is not defined $\Rightarrow M(\delta(M))\downarrow$.

How to prove undecidability of HP on any strongly generic set?

Let S be a strongly generic set of programs. Suppose HP is decidable on S, then

$$halt(x) = \begin{cases} 1, & \text{if } x = \delta(M) \text{ and } M(x)\downarrow, \\ 0, & \text{if } x = \delta(M) \text{ and } M(x)\uparrow. \end{cases}$$

is computable function on $\delta(S)$. Hence the function

$$diag(x) = \begin{cases} \text{not def}, & \text{if } x = \delta(M) \text{ and } M(x)\downarrow, \\ 0, & \text{if } x = \delta(M) \text{ and } M(x)\uparrow. \end{cases}$$

is computable on $\delta(S)$ and computed by some machine M. To get a contradiction we must give M the input $\delta(M)$.

Question 6 *Should $\delta(M)$ belong to $\delta(S)$? Should M be in S?*

Lemma 1 *For any computable function f the set $C(f)$ of all machines computing f is not strongly negligible.*

Idea of proof. M has k states and computes f. M^* has $n > k$ states and program with the same transition rules as in M for first k states and arbitrary rules for $n - k$ other states:

$$\text{fixed } 3k \text{ rules } \begin{cases} (q_1, 0) \to \ldots, \\ \ldots \\ (q_k, \Box) \to \ldots, \end{cases}$$

$$\text{arbitrary } 3(n-k) \text{ rules } \begin{cases} (q_{k+1}, 0) \to \ldots, \\ \ldots \\ (q_n, \Box) \to \ldots. \end{cases}$$

M^* computes f. A is the set of all such M^*.

$$\frac{|C(f) \cap P_n|}{|P_n|} \geq \frac{|A \cap P_n|}{|P_n|} =$$

$$= \frac{(6(n+1))^{3(n-k)}}{(6(n+1))^{3n}} = \frac{1}{(6(n+1))^{3k}}.$$

So $C(f)$ is not strongly negligible.

Returning to HP.

- $C(diag)$ is not strongly negligible

- $P \setminus S$ is strongly negligible.

$\Rightarrow$ there is a machine M computing $diag$ such that $M \in S$. That is all that we need to end proof of main theorem.

The end. Thank you.