Definable subsets in a free group

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November 11

Abstract

We give a description of definable subsets in a free non-abelian group F that follows from our work on the Tarski problems. As a corollary we show that proper non-abelian subgroups of F are not definable (Malcev's problem) and prove Bestvina and Feighn's result that definable subsets in a free group are either negligible or co-negligible.

Let F be a free group with finite basis. We consider formulas in the language L_A that contains generators of F as constants. Notice that in the language L_A every finite system of equations is equivalent to one equation (this is Malcev's result) and every finite disjunction of equations is equivalent to one equation (this is attributed to Gurevich).

Quantifire Elimination

Theorem

(Sela,Kh,Miasn) Every formula in the theory of F is equivalent to the boolean combination of AE-formulas.

Furthermore, a more precise result holds.

Theorem

Every definable subset of F is defined by some boolean combination of formulas

$$\exists X \forall Y (U(P, X) = 1 \wedge V(P, X, Y) \neq 1), \tag{1}$$

where X, Y, P are tuples of variables.

Definition

A *piece* of a word $u \in F$ is a non-trivial subword that appears in two different ways.

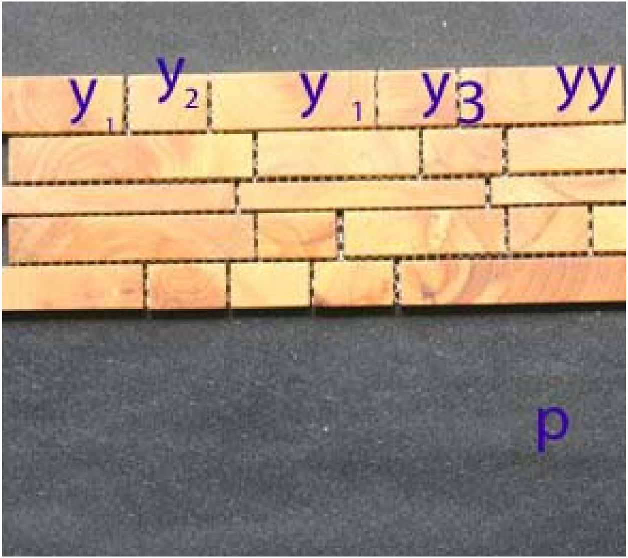
Definition

A proper subset P of F admits parametrization if it is a set of all words p that satisfy a given system of equations (with coefficients) without cancellations in the form

$$p \mathrel{\mathring{=}} w_t(y_1, \ldots, y_n), t = 1, \ldots, k, \tag{2}$$

where for all $i = 1, \ldots, n$, $y_i \neq 1$, each y_i appears at least twice in the system and each variable y_i in w_1 is a piece of p.

The empty set and one-element subsets of F admit parametrization.


y1
y2
y1
y3
yy
p

Definition

(BF) A subset P of F is *negligible* if there exists $\epsilon > 0$ such that all but finitely many $p \in P$ have a piece such that

$$\frac{length(piece)}{length(p)} \geq \epsilon.$$

A complement of a negligible subset is *co-negligible*.

Bestvina and Feighn stated that in the language without constants every definable set in F is either negligible or co-negligible. They also proved that 1) Subsets of negligible sets are negligible.
2) Finite sets are negligible.
3) A set S containing a coset of a non-abelian subgroup G of F cannot be negligible
4) A proper non-abelian subgroup of F is neither negligible nor co-negligible.
5) The set of primitive elements of F is neither negligible nor co-negligible if $rank(F) > 2$.

Proof.

3) If $x, y \in G$ and $[x, y] \neq 1$, then the infinite set $\{fxyxy^2x \dots xy^ix,\ i \in \mathbb{N}\}$ is not negligible .
Statement 4) follows from 3).
5) Let a, b, c be three elements in the basis of F and denote $F_2 = F(a, b)$ The set of primitive elements contains cF_2, and the complement contains $< [a, b], c^{-1}[a, b]c >$.

□

Lemma

A set P that admits parametrization is negligible.

Proof.

Let m be the length of word w_1 (as a word in variables y_i's and constants). The set P is negligible with $\epsilon = 1/m$. □

It follows from [K.,M.: Imp] that every E formula in the language L_A is equivalent to $\exists X(U(X,Y)=1 \wedge V(X,Y) \neq 1)$, where X, Y are families of variables. In our case Y consists of one variable p, and the formula takes form $\exists X(U(X,p)=1 \wedge V(X,p) \neq 1)$.

Theorem

Suppose an E-set P is not the whole group F and is defined by the formula

$$\psi(p) = \exists Y U(Y,p) = 1,$$

then it is a finite union of sets admitting parametrization.

Corollary

Suppose an E-set P is defined by the formula

$$\psi_1(p) = \exists Y(U(Y,p) = 1 \wedge V(Y,p) \neq 1).$$

If the positive formula $\psi(p) = \exists Y(U(Y,p) = 1$ does not define the whole group F, then P is negligible, otherwise it is empty (therefore, negligible) or co-negligible.

Proof.

If $\psi(p)$ does not define the whole group F, then $\psi_1(p)$ defines a subset of the negligible set and is negligible.

Suppose now that $\psi(p)$ defines the whole group. Then $\psi_1(p)$ is equivalent to $\psi_2(p) = \exists Y V(Y, p) \neq 1$. Suppose it defines a non-empty set.

Consider $\neg\psi_2(p) = \forall Y V(Y, p) = 1$. This is equivalent to a system of equations in p, that does not define the whole group, therefore it defines a finite union of sets admitting parametrization and this set is negligible. In this case P is co-negligible. □

Theorem

For every definable subset P of F, P or its complement $\neg P$ is a subset of a finite union of sets admitting parametrization.

Corollary

(B,F) Every definable subset of F in the language with constants (and, therefore, in the language without constants) is either negligible or co-negligible.

This implies the solution to Malcev's problem.

Corollary

Proper non-abelian subgroups of F are not definable.

Corollary

The set of primitive elements of F is not definable if $\mathrm{rank}(F) > 2$.

Cut Equations

Cut Equations

Definition

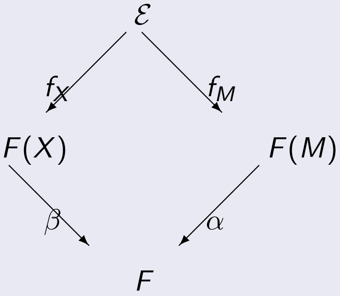
A cut equation $\Pi = (\mathcal{E}, M, X, f_M, f_X)$ consists of a set of intervals $\mathcal{E}$, a set of variables M, a set of parameters X, and two labeling functions

$$f_X : \mathcal{E} \to F[X], \quad f_M : \mathcal{E} \to F[M].$$

For an interval $\sigma \in \mathcal{E}$ the image $f_M(\sigma) = f_M(\sigma)(M)$ is a reduced word in variables $M^{\pm 1}$ and constants from F, we call it a *partition* of $f_X(\sigma)$.

Definition

A solution of a cut equation $\Pi = (\mathcal{E}, f_M, f_X)$ with respect to an F-homomorphism $\beta : F[X] \to F$ is an F-homomorphism $\alpha : F[M] \to F$ such that: 1) for every $\mu \in M$ $\alpha(\mu)$ is a reduced non-empty word; 2) for every reduced word $f_M(\sigma)(M)$ $(\sigma \in \mathcal{E})$ the replacement $m \to \alpha(m)$ $(m \in M)$ results in a word $f_M(\sigma)(\alpha(M))$ which is a reduced word as written and such that $f_M(\sigma)(\alpha(M))$ is graphically equal to the reduced form of $\beta(f_X(\sigma))$; in particular, the following diagram is commutative.



Cut Equations

Theorem

Let $S(X, Y, A)) = 1$ *be a system of equations over* $F = F(A)$. *Then one can effectively construct a finite set of cut equations*

$$\mathcal{CE}(S) = \{\Pi_i \mid \Pi_i = (\mathcal{E}_i, f_{X_i}, f_{M_i}), i = 1 \ldots, k\}$$

and a finite set of tuples of words $\{Q_i(M_i) \mid i = 1, \ldots, k\}$ *such that:*

1. for any solution (U, V) *of* $S(X, Y, A) = 1$ *in* $F(A)$, *there exists a number* i *and a tuple of words* $P_{i,V}$ *such that the cut equation* $\Pi_i \in \mathcal{CE}(S)$ *has a solution* $\alpha : M_i \to F$ *with respect to the* F*-homomorphism* $\beta_U : F[X] \to F$ *which is induced by the map* $X \to U$. *Moreover,* $U = Q_i(\alpha(M_i))$, *the word* $Q_i(\alpha(M_i))$ *is reduced as written, and* $V = P_{i,V}(\alpha(M_i))$;

2. for any $\Pi_i \in \mathcal{CE}(S)$ *there exists a tuple of words* $P_{i,V}$ *such that for any solution (group solution)* (β, α) *of* Π_i *the pair* (U, V), *where* $U = Q_i(\alpha(M_i))$ *and* $V = P_{i,V}(\alpha(M_i))$, *is a solution of* $S(X, Y) = 1$ *in* F.

Proof of the Theorem

In our case formula (1) has form

$$\exists X \forall Y (U(p, X) = 1 \wedge V(p, X, Y) \neq 1), \tag{3}$$

where X, Y, P are tuples of variables.
If the E-set defined by $\exists X(U(p, X) = 1$ is not the whole group, then the set P defined by the formula (3) is a subset of finite union of sets admitting parametrization.
Suppose now that the set defined by $\exists X(U(p, X) = 1$ is the whole group, then formula (3) is equivalent to

$$\exists X \forall Y V(X, Y, p) \neq 1.$$

Suppose it does not define the empty set. Then the negation is

$$\phi_1(p) = \forall X \exists Y V(X, Y, p) = 1$$

and defines $\neg P$.

Proof of the Theorem

Lemma

Formula

$$\theta(P) = \forall X \exists Y V(X, Y, P) = 1$$

in F in the language L_A is equivalent a positive E-formula $\exists Z U(P, Z) = 1$.

Since $\neg P \neq F$, by this lemma, it must be a finite union of sets admitting parametrization.

Negligible subsets are negligible

Definition

Recall that in complexity theory $T \subseteq F(X)$ is called generic if

$$\rho_n(T) = \frac{|T \cap B_n(X)|}{|B_n|} \to 1, \ \textit{if} \ \ n \to \infty,$$

where $B_n(X)$ is the ball of radius n in the Cayley graph of $F(X)$. A set is negligible is its complement is generic.

Negligible subsets are negligible

Theorem

Negligible sets are negligible .

Thanks!