Rigid Solvable Groups

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Algebraic geometry over groups and other algebraic systems.
G.Baumslag, O.Kharlampovich, A.Myasnikov, B.Plotkin, V.Remeslennikov.

Introduction to algebraic geometry over groups:

G.Baumslag, A.Myasnikov and V.Remeslennikov, Algebraic geometry over groups. I. Algebraic sets and ideal theory, J. Algebra, 219, N 1 (1999), 16-79.
A.Myasnikov and V.Remeslennikov, Algebraic geometry over groups. II. Logical foundations, J. Algebra, 234, N 1 (2000), 225-276.

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Our talk based on following papers:

C.K.Gupta, N.S.Romanovskiy, *The property of being equationally Noetherian for some soluble groups*, Algebra and Logic, 46(1), 2007, pp. 28-36.
N.S.Romanovskiy, *Divisible rigid groups*, Algebra and Logic, 47(6), 2008, pp. 426-434.
N.S.Romanovskiy, *Equational Noetherianess of rigid solvable groups*, Algebra and Logic, 48(2), 2009, pp. 147-160.
N.S.Romanovskiy, *Irreducible algebraic sets over divisible decomposed rigid groups*, Algebra and Logic, 48(6), 2009, pp. 449-464.
A.Myasnikov, N.Romanovskiy, *Krull dimension of solvable groups*, J.Algebra, 324 (10), 2010, pp. 2814-2831.
N.S.Romanovskiy, *Coproducts of rigid groups*, Algebra and Logic, 49 (6), 2010, pp. 539-550.
A.Myasnikov, N.S.Romanovskiy, *On universal theories of rigid solvable groups*, submitted for publication.
N.S.Romanovskiy, *On representations of rigid solvable groups by defining relations*, submitted for publication.

$G \triangleright A$, A is abelian. G acts by conjugation: $a \to a^g = g^{-1}ag$.
G/A acts, A is a right $\mathbb{Z}[G/A]$-module.
$u = \alpha_1\overline{g}_1 + \ldots + \alpha_n\overline{g}_n \in \mathbb{Z}[G/A]$, $a^u = (a^{g_1})^{\alpha_1} \cdot \ldots \cdot (a^{g_n})^{\alpha_n}$.

Definition

m-rigid group G: there is a normal series

$$G = G_1 > G_2 > \ldots > G_m > G_{m+1} = 1,$$

G_i/G_{i+1} are abelian and considering as right $\mathbb{Z}[G/G_i]$-modules have no torsion.

Why rigid? - this series is unique. Given group G is solvable of length exactly m.

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1) Free solvable group is rigid, rigid series consists of commutator subgroups.
2) $W = A_m \wr (A_{m-1} \wr (\ldots \wr A_1)\ldots)$, where A_i are free abelian groups.

Subgroups of rigid groups are rigid too: $G \geqslant H,\ H_i = H \cap G_i$. Corresponding series for H may be shorter.

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G-group, G-subgroups, G-homomorphism, ...

$F = G * \langle x_1, \ldots, x_n \rangle,\ x = (x_1, \ldots, x_n).$
Equation $v(x) = 1,\ v(x) \in F.$
F is a group of all equations.
$\{v_i(x) = 1\ (i \in I)\}$, set of solutions $S \subseteq G^n$ is called algebraic set.

Annihilator of S: $I(S) = \{v(x) \in F \mid v(S) = 1\}$.
Coordinate group of S: $\Gamma(S) = F/I(S)$.
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In particular, if $G \in \mathfrak{M}$, then $\Gamma(G^n) \in \mathfrak{M}$.
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The intersection of algebraic sets $\bigcap S_i$ is algebraic, but the union $S_1 \cup S_2$ is not in general case and very often when G is solvable.

Zariski topology on G^n: take the algebraic sets as a sub-basis for the closed sets.

The topology is Noetherian, if every properly descending chain of closed subsets is finite. In this case $S = S_1 \cup \ldots \cup S_k$, $S_i \not\subseteq S_j$, S_i are irreducible components.

We say that given group is equationally Noetherian (EN) if for any n arbitrary system of equations on $x_1, \ldots, x_n$ over this group is equivalent to some finite subsystem.

Proposition 1 *G is EN $\Leftrightarrow$ for any n Zariski topology on G^n is Noetherian. In this case irreducible sets are algebraic.*

Hard to study algebraic geometry over given group without last property. So, to be equationally Noetherian group is necessary condition for good algebraic geometry.

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2) Linear groups, in particular, free groups.

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Example 1.
$A = \langle a_1, a_2, \ldots, b_1, b_2, \ldots \ |\mathfrak{N}_2,\ [b_1, a_1] = 1,$
$[b_2, a_1] = [b_2, a_2] = 1, \ldots, [b_n, a_1] = \ldots = [b_n, a_n] = 1, \ldots\rangle$
A system of equations $\{[x, a_i] = 1\}$ isn't equivalent to a finite subsystem.

Example 2.
$H = \langle c, d\rangle$ is a free centre-by-metabelian group, $[H, H]$ contains a free nilpotent group of class 2 with a countable basis $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$.
$G = \langle c, d \mid \ldots\rangle \geqslant A.$

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Theorem 1 (R, 2009)

Rigid groups are equationally Noetherian.

For free solvable groups and f.g. rigid groups it was proved by GR in 2007.

Main purpose of algebraic geometry over group: to describe algebraic sets. For EN groups we have two problems.
1. To describe irreducible sets.
2. To define when the union $S_1 \cup \ldots \cup S_k$ of irreducible algebraic sets is algebraic.

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Let H_1, H_2 be G-groups.
H_2 separates H_1: $1 \neq a \in H_1,\ \varphi : H_1 \to H_2,\ a\varphi \neq 1$.
H_2 discriminates H_1:
1) $\{a_1, \dots, a_n\},\ 1 \neq a_i \in H_1,\ \varphi : H_1 \to H_2,\ a_i\varphi \neq 1$,
or 2)$\{a_1, \dots, a_n\},\ a_i \neq a_j \in H_1,\ \varphi : H_1 \to H_2,\ a_i\varphi \neq a_j\varphi$.

Proposition 2 *Let $H = \langle G,\ y_1, \dots, y_n\rangle$.*
1) H is a coordinate group of an algebraic set $S \subseteq G^n$ on $y_1, \dots, y_n$ $\Leftrightarrow$ H is G-separated by G.
2) If $G \in EN$ then H is a coordinate group of an irreducible algebraic set $S \subseteq G^n$ on $y_1, \dots, y_n \Leftrightarrow H$ is G-discriminated by G.

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Let H be G-group. Denote by $U_G(H)$ the universal theory ($\forall$-theory) of H with constants from G.
$\forall x_1, \ldots, x_n \Phi(x)$, $\Phi(x): \ \bigvee, \bigwedge \ v(x) = 1, \ v(x) \neq 1.$

Proposition 3 *Let H_1, H_2 be G-groups which are EN by equations with constants from G. Then $U_G(H_1) = U_G(H_2) \Leftrightarrow H_1$ is locally discriminated by H_2 and H_2 is locally discriminated by H_1.*

Proposition 4 *Let $H = \langle G, \ y_1, \ldots, y_n \rangle$ be EN by equations with constants from G. Then H is a coordinate group of an irreducible algebraic set $S \subseteq G^n$ on $y_1, \ldots, y_n \Leftrightarrow U_G(H) = U_G(G)$.*

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In the papers

V.N.Remeslennikov, N.S.Romanovskiy, Irreducible algebraic sets in metabelian groups, Algebra and Logic, 44(5), 2005, pp. 336-347,
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$\delta_1 = x_1,\ \delta_2 = [x_1, x_2],\ \delta_3 = [[x_1, x_2], [x_3, x_4]], \ldots$
define corresponding commutator subgroups.
Take $a_m = \delta_m(\ldots) \neq 1,\ a_{m-1} = \delta_{m-1}(\ldots) \notin G_m,\ a_{m-2} = \delta_{m-1}(\ldots) \notin G_{m-1}, \ldots$
G_m is a centralizer of a_m, G_{m-1} is a centralizer of a_{m-1} modulo G_m, G_{m-2} is a centralizer of a_{m-2} modulo $G_{m-1}, \ldots$.
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G_m is a centralizer of a_m, G_{m-1} is a centralizer of a_{m-1} modulo G_m, G_{m-2} is a centralizer of a_{m-2} modulo $G_{m-1}, \ldots$.
$[a_m, x] = a_m^{-1} x^{-1} a_m x = a_m^{x-1} = 1 \Leftrightarrow x \in G_m$.

Some facts about rings.
Right Ore domain R: no zero divisors and for any $a, b \in R$ there is a nontrivial solution of equation $ax = by$.
$\{ab^{-1} \mid a, b \in R, b \neq 0\}$ is the right Ore skew field of fractions.
If R is also a left Ore domain then the right Ore skew field of fractions = the left Ore skew field of fractions.

Proposition 5 *If G is a solvable torsion free group then the group ring $\mathbb{Z}G$ is an Ore domain, so possible to consider the Ore skew field of fractions which we denote by $Q(G)$.*

It follows from P.H.Kropholler, P.A.Linnell and J.A.Moody, Applications of a new K-theoretic theorem to soluble group rings, Proc. Amer. Math. Soc., 104 (1988), 675-684.

Some facts about rings.
Right Ore domain R: no zero divisors and for any $a, b \in R$ there is a nontrivial solution of equation $ax = by$.
$\{ab^{-1} \mid a, b \in R, b \neq 0\}$ is the right Ore skew field of fractions.
If R is also a left Ore domain then the right Ore skew field of fractions = the left Ore skew field of fractions.

Proposition 5 *If G is a solvable torsion free group then the group ring $\mathbb{Z}G$ is an Ore domain, so possible to consider the Ore skew field of fractions which we denote by $Q(G)$.*

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Definition

Rigid group G is called divisible if any factor $T_i = G_i/G_{i+1}$ is a divisible module over the ring $\mathbb{Z}[G/G_i]$ or, in other words, T_i is a vector space over skew field of fractions $Q(G/G_i)$.

Let $\alpha_1, \dots, \alpha_m$ be nonzero cardinalities. Construct group $M(\alpha_1, \dots, \alpha_m)$ by induction. $M(\alpha_1)$ is a direct sum of α_1 copies of $\mathbb{Q}$. $A = M(\alpha_1, \dots, \alpha_{m-1})$. Let T be a vector space with a basis of cardinality α_m over the skew field $Q(A)$. Then set

$$M(\alpha_1, \dots, \alpha_m) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

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Theorem 2 (R, 2009)

Arbitrary m-rigid group can be embedded into some decomposed divisible m-rigid group.

Finitely generated rigid groups are exactly finitely generated subgroups of iterated wreath products of free abelian groups $W = A_n \wr (A_{n-1} \wr (\ldots \wr A_1)\ldots)$.

$G \geqslant H,\ H_i = H \cap G_i,\ H_i/H_{i+1} \leqslant G_i/G_{i+1},\ \mathbb{Z}[H/H_i] \leqslant \mathbb{Z}[G/G_i]$. We say that H is embedded into G with preserving linear independence, if any system elements of H_i/H_{i+1} linear independent over the ring $\mathbb{Z}[H/H_i]$ has to be linear independent over the ring $\mathbb{Z}[G/G_i]$.

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Let G be a m-rigid subgroup of divisible rigid group D. Then there is a minimal divisible subgroup containing G, let it be $\overline{G}$ = divisible closure of G in D. This subgroup $\overline{G}$ is m-rigid and $\overline{G}_i/\overline{G}_{i+1}$ is generated by the set G_i/G_{i+1} as a vector space over $Q(\overline{G}/\overline{G}_i)$.

Natural question: Let G_1 and G_2 be two divisible closures of G, are they G-isomorphic?
NO, in general case, but YES with adding condition:

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For given m-rigid group G there is such divisible closure $\widehat{G}$ that G is embedded into $\widehat{G}$ with preserving linear independence. We call $\widehat{G}$ divisible completion of G. Any two divisible completions of G are G-isomorphic.

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$T_i = G_i/G_{i+1}$ is a torsion free module over the ring $\mathbb{Z}[G/G_i]$.
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T_i embeds into the right vector space
$V_i = T_i \otimes_{\mathbb{Z}[G/G_i]} Q(G/G_i)$, $r_i(G) = \dim V_i$.

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Let G be m-rigid group, $S \subseteq G^n$ be an irreducible algebraic set, $\Gamma = \Gamma(S)$.
1) *Then Γ is m-rigid.*
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3) *Inequality $d_i(S) \leq n$ holds.*

The tuple $d(S) = (d_1(S), \ldots, d_p(S))$ is called a dimension of irreducible algebraic set S. For finitely generated G all ranks $r_i(G)$ and $r_i(\Gamma)$ are finite and so $d(S) = r(\Gamma) - r(G)$.

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Remind that topological dimension of given topological space by definition is equal to supremum of lengths of chains $S_1 > S_2 > \ldots > S_m$ irreducible subsets.

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If G be a m-rigid group then the topological dimension of the space G^n is finite and doesn't exceed the number $(n+1)^m$.

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Let $M = M(\alpha_1, \ldots, \alpha_m)$. Then finitely generated M-group G is a coordinate group of some irreducible algebraic set over M if and only if G is m-rigid and M is embedded into G with preserving linear independence.

We deduce the theorem 8 from following statement.

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Let a group G contain $M = M(\alpha_1, \ldots, \alpha_m)$ as a subgroup. Then G is M-universally equivalent to M if and only if G is m-rigid and M is embedded into G with preserving linear independence.

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Malcev proved that free solvable group of class ≥ 2 has undecidable elementary theory. The universal theory of free metabelian group was studied by Chapuis, Remeslennikov and Stohr, in particular, it is decidable. Free solvable groups of given class ≥ 3 and different ranks are universally equivalent too and Chapuis proved that their universal theory is undecidable if the universal theory of rational numbers is undecidable (10-th Hilbert problem for rational numbers). Nevertheless, Chapuis proved that the the universal theory of an iterated wreath product of of free abelian groups is decidable.

We construct a recursive system of ∀-axioms which define m-rigid groups in the class of all m-soluble groups.

Let F denote a free solvable group of class m, G denote an arbitrary m-rigid group, W denote the iterated wreath product of of m cyclic groups. For corresponding universal theories we prove conclusions

$$\mathcal{A}(F) \supseteq \mathcal{A}(G) \supseteq \mathcal{A}(W).$$

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We prove that the universal theory of W with constants from W is undecidable.
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Theorem 9 (MR). *The universal theory of the group $M = M(\alpha_1, \ldots, \alpha_m)$ with constants from M is decidable.*

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Isomorphic subgroups of given rigid group are not the same.

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i = 0, 1$.

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m-graduated rigid group G with graduation ε: there is a normal series

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m-rigid group has only one m-graduation: $(1, 1, \ldots, 1, 1)$.

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Theorem 11 (R, 2010). *Let G and H be two m-graduated rigid groups. Then there is m-graduated rigid group $G \circ H$, which is called m-rigid product of G and H, with following properties.*
1) *Groups G and H are subgroups of $G \circ H$ and generate it.*
2) *Any homomorphisms*

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of m-graduated rigid groups continue to homomorphism

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Let $F_1, \ldots, F_n$ be infinite cyclic groups with m-graduation $(1, 0, \ldots, 0)$). Then their m-rigid product $F_1 \circ \ldots \circ F_n$ is a free m-solvable group of the rank n.
Let A and B be rigid groups with m-graduations $(0, \ldots, 0, 1)$, and $(1, \ldots, 1, 0)$. Then the product $A \circ B$ is isomorphic to $A \wr B$.

Theorem 12 (R, 2010). *Let G be m-rigid group and F be free m-solvable group (for rank 1 with m-graduation $(1, 0, \ldots, 0)$). Then m-rigid product $G \circ F$ is G-discriminated by G.*

Corollary. *Let $\{x_1, \ldots, x_n\}$ be a basis of the group F in the theorem. Then $G \circ F$ is a coordinate group of the affine space G^n on $x_1, \ldots, x_n$ and this space is irreducible.*

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Σ_m = all $\leq m$-rigid groups. Arbitrary n-generated group of Σ_m is a factor group of a free solvable group $F_{m,n}$ of length m with a basis $\{x_1, \ldots, x_n\}$ by some ideal. How can we represent groups in Σ_m by defining relations? $R = R(x_1, \ldots, x_n)$ — some set of group words on $x_1, \ldots, x_n$. In a classic case the group $\langle x_1, \ldots, x_n \mid R\rangle$ is a factor group of a free group by the least normal subgroup containing R. In our case not always there is a least ideal of $F_{m,n}$ containing R. Example: $m = 2,\ n = 3,\ R = \{[x_1, x_2]^{x_3 - 1}\}$. If $G \in \Sigma_2$ is generated by x_1, x_2, x_3, and $[x_1, x_2]^{x_3 - 1} = 1$ then or $[x_1, x_2] = 1$, or $x_3 \in G_2$. First group is defined in the variety $\mathfrak{A}^2$ by generators x_1, x_2, x_3 and relation $[x_1, x_2] = 1$, here $x_3 \notin G_2$. Second group is defined by relations $[x_3, F'_{2,3}] = 1$, here $[x_1, x_2] \neq 1,\ x_3 \in G_2$. And there isn't a group of Σ_2 with relation $[x_1, x_2]^{x_3 - 1} = 1$ which covers both groups.

$\Sigma_m(R)$ = all groups of Σ_m generated by $x_1, \ldots, x_n$ with relations R. Maximal group = there is no proper covering in $\Sigma_m(R)$. Theorem 13 $\Rightarrow$ for any group of $\Sigma_m(R)$ there is a maximal covering. Any maximal groups of $\Sigma_m(R)$ possible to understand as a group defining in Σ_m by generators $x_1, \ldots, x_n$ and relations R.

Theorem 14 (R)

For arbitrary R the set $\Sigma_m(R)$ contains only finite number of maximal groups.

R is called complete set of defining relations if $\Sigma_m(R)$ contains only single maximal group, so this group is defined by relations R uniquely.

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For arbitrary R the set $\Sigma_m(R)$ contains only finite number of maximal groups.

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For arbitrary finite set of relations $R = R(x_1, \ldots, x_n)$ there is an effectively procedure of constructing of some finite set $\Omega_m(R)$ of canonical representations on generators $x_1, \ldots, x_n$ of groups of $\Sigma_m(R)$ such that $\Omega_m(R)$ contains all maximal groups of $\Sigma_m(R)$.

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Theorem 1 (R, 2009)

Rigid groups are equationally Noetherian.

The group $M(\alpha_1, \ldots, \alpha_m)$ is constructed by induction. $M(\alpha_1)$ is a direct sum of α_1 copies of $\mathbb{Q}$. $A = M(\alpha_1, \ldots, \alpha_{m-1})$. Let T be a vector space with a basis of cardinality α_m over the skew field $Q(A)$. Then set

$$M = M(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

M is a semidirect product $A_1 A_2 \ldots A_m$ of abelian groups A_i, $A_m = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$. $M_i = A_i A_{i+1} \ldots A_m$ are members of rigid series.

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Let $1 \neq a_i \in A_i$. Note that A_i is exactly the centralizer of a_i in M, so $A_i = \{x \in M \mid [a_i, x] = 1\}$.

Consider following set of variables $X = \{x_{ij} \mid i = 1, \ldots, n;\ j = 1, \ldots, m\}$ and its subset $X' = \{x_{ij} \mid i = 1, \ldots, n;\ j = 1, \ldots, m-1\}$. We suppose that values of the variable x_{ij} belong to A_j. Call x_{ij} special variables. Possible to consider them as usual variables with adding conditions $[x_{ij}, a_j] = 1$. Last equations define an algebraic set which we denote by Ω. We define also special algebraic equations and special algebraic sets (subsets of Ω). Also Ω possible to identify with M^n.

Let $x_1 = x_{11}x_{12} \ldots x_{1m}, \ldots, x_n = x_{n1}x_{n2} \ldots x_{nm}$. We see that any elements of M can be values of x_i. So usual equations on $x_1, \ldots, x_n$ can be realized as special equations on X.

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Theorem 1' (R, 2009)

1) *The group M is EN by special equations on X.*
2) *Let S be a special irreducible algebraic subset of M^n and $\varphi : \Gamma(S) \to M$ be arbitrary specialization. Then $\ker \varphi$ is separated by nilpotent torsion free groups.*

For our purpose we need only statement 1), but the proof by induction will use statement 2).
By induction suppose that for $A = M(\alpha_1, \ldots, \alpha_{m-1})$ corresponding statements hold.

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By induction suppose that for $A = M(\alpha_1, \ldots, \alpha_{m-1})$ corresponding statements hold.

Now we construct a group of special equations $M[X]$. Let the group $A_1[x_{11}, \ldots, x_{n1}]$ be equal a direct product of A_1 and a free abelian group with a basis $\{x_{11}, \ldots, x_{n1}\}$. Suppose by induction the group $B = A[X']$ is costructed and let it be generated by its subgroup A and the set X' and there be a decomposion of B as semidirect product $B_1 B_2 \ldots B_{m-1}$, where the abelian group B_j contains A_j and $x_{ij} \in B_j$. Suppose that any mapping $x_{ij} \to a_{ij} \in A_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, m-1)$ possible to continue to an A-epimorphism $B \to A$.

Consider a direct sum of the module $T \otimes_{\mathbb{Z}A} \mathbb{Z}B$ and the right free $\mathbb{Z}B$-module with the basis $\{x_{1m}, \ldots, x_{nm}\}$. Set

$$M[X] = \begin{pmatrix} B & 0 \\ T \otimes_{\mathbb{Z}A} \mathbb{Z}B + x_{1m} \cdot \mathbb{Z}B + \ldots + x_{nm} \cdot \mathbb{Z}B & 1 \end{pmatrix},$$

here we indentify the element x_{ij} for $j < m$ with the matrix $\begin{pmatrix} x_{ij} & 0 \\ 0 & 1 \end{pmatrix}$, and the element x_{im} with the matrix $\begin{pmatrix} 1 & 0 \\ x_{im} & 1 \end{pmatrix}$. The group $M[X]$ is generated by its subgroup M and the set X and $M[X] = B \cdot B_m$, where the subgroup B_m is isomorphic to the additive group of the module

$$T \otimes_{\mathbb{Z}A} \mathbb{Z}B + x_{1m} \cdot \mathbb{Z}B + \ldots + x_{nm} \cdot \mathbb{Z}B.$$

We prove that any mapping $x_{ij} \to a_{ij} \in A_j$ possible to continue to an M-epimorphism $M[X] \to M$. It means that $M[X]$ is a group of special equations on X over M.

An equation of the type $u(X') = 0$, where $u(X') \in \mathbb{Z}B$, is called a group ring equation, for this equation we find solutions with restriction $x_{ij} \in A_j$. It is important here: arbitrary group ring equation is equivalent to some disjunction of finite number of finite systems of group equations.
For example the equation $V_1 + V_2 + V_3 - V_4 - 2V_5 = 0$, where $V_i \in A[X']$, is equivalent to
$((V_1 = V_4) \wedge (V_2 = V_3 = V_5)) \vee ((V_2 = V_4) \wedge (V_1 = V_3 = V_5)) \vee$
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So, a group ring equation defines a closed subset in A^n. Easy to note that any closed subset of A^n can be defined by some group ring equation.

Possible to realize arbitrary group ring equation $u(X') = 0$ as a group equation $v(X) = 1,\ v(X) \in M[X]$: take some nontrivial element $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in M$ and set $v(X) = \begin{pmatrix} 1 & 0 \\ t \cdot u(X') & 1 \end{pmatrix}$.

We prove also that if L is an irreducible subset of M^n then the closure of its projection on A^n is irreducible too.

Now come immediately to the proof of the theorem 1'.

Consider an arbitrary system of special equations on X over M :

$$\{w_j(X) = 1,\ j \in J. \tag{1}$$

Let

$$w_j(X) = \begin{pmatrix} f_j(X') & 0 \\ v_j(X) & 1 \end{pmatrix}.$$

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The system (1) is equivalent to the union of two systems:

$$\{f_j(X') = 1,\ j \in J. \tag{2}$$

$$\{v_j(X) = 0,\ j \in J. \tag{3}$$

By induction the system (2) is equivalent to some its finite subsystem. If (2) has no solutions then some finite subsystem of (1) has no solutions. So, suppose that the system (2) defines nonempty closed subset in A^n.

Suppose we have a counterexample to the statement 1) of our theorem. Then there is nonempty closed subset S of A^n such that the system (3) with condition $X' \in S$ isn't equivalent to finite subsystem. We can suppose that S is minimal with this property, then it is irreducible.

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So, for any closed proper nonempty subset $P \subset S$ the system (3) with condition $X' \in P$ is equivalent to some its finite subsystem, but with condition $X' \in S$ it isn't equivalent to finite subsystem.

Let C denote the coordinate group $\Gamma(S)$. This group is generated by A and the images of the elements $x_{ij} \in X'$ which we will denote by the same symbols. As this group is discriminated by A then it is soluble torsion free. Consider the skew field of fractions $Q(C)$ and its subring R generated by $Q(A)$ and X'. We remember that T is a right vector space over $Q(A)$. Embed it into the free R-module TR with the same basis. Consider a direct sum of TR and a free right $\mathbb{Z}C$-module with the basis $\{x_{1m}, \ldots, x_{nm}\}$. Then take a group

$$D = \begin{pmatrix} C & 0 \\ TR + x_{1m} \cdot \mathbb{Z}C + \ldots + x_{nm} \cdot \mathbb{Z}C & 1 \end{pmatrix}$$

and identify x_{ij} for $j < m$ with $\begin{pmatrix} x_{ij} & 0 \\ 0 & 1 \end{pmatrix}$, and x_{im} with $\begin{pmatrix} 1 & 0 \\ x_{im} & 1 \end{pmatrix}$.

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We prove that the group D is a group of special equations on X over M with condition $X' \in S$.

Therefore possible to suppose that the left parts of equations (3) belong to the module $TR + x_{1m} \cdot \mathbb{Z}C + \ldots + x_{nm} \cdot \mathbb{Z}C$.

Embed this module to the right vector spase $T \cdot Q(C) + x_{1m} \cdot Q(C) + \ldots + x_{nm} \cdot Q(C)$ and denote by V a subspase generated by v_j $(j \in J)$.

We prove that $V \cap T \cdot Q(C) = 0$, therefore the projection V onto the spase $x_{1m} \cdot Q(C) + \ldots + x_{nm} \cdot Q(C)$ is injective. Then $\dim V \leq n$. Let $\{v_1, \ldots, v_r\}$ be a basis of V. By elementary transformations over the ring $\mathbb{Z}C$ we make this system generalized diagonal with the same element u in diagonal.

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Possible suppose that

$$v_1 = x_{1m}u + v_1', \ldots, v_r = x_{rm}u + v_r',\ v_i' \in TR + x_{r+1,m}\cdot\mathbb{Z}C + \ldots + x_{nm}\cdot\mathbb{Z}C.$$

Consider the ring equation $u(X') = 0$ with condition $X' \in S$. It defines proper (may be empty) closed subset of S, let it be P. If P isn't empty there is a finite subsystem Σ_1 of the system (3) which is equivalent to (3) with condition $X' \in P$. For empty P suppose Σ_1 is empty.

Denote by Σ_2 the system $v_1(X) = 0, \ldots, v_r(X) = 0$.

We will prove that with condition $X' \in S \setminus P$ the system (3) is equivalent to Σ_2.

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Let

$$a = (a_{ij} \in A_{ij} \mid i = 1, \ldots, n;\ j = 1, \ldots, m)$$

be a solution of the system Σ_2 and

$$a' = (a_{ij} \mid i = 1, \ldots, n;\ j = 1, \ldots, m-1) \in S \setminus P.$$

Consider arbitrary equation $v_j(X) = 0$ of (3). We have to prove that $X = a$ is its solution.

Let

$$v_j = x_{1m}u_1 + \ldots + x_{rm}u_r + w,\ u_i \in \mathbb{Z}C,\ w \in TR + x_{r+1,m} \cdot \mathbb{Z}C + \ldots + x_{nm} \cdot \mathbb{Z}C.$$

As $v_j \in V$ and $\{v_1, \ldots, v_r\}$ is a basis of V, then

$$v_j(X) = v_1(X) \cdot u(X')^{-1} \cdot u_1(X') + \ldots + v_r(X) \cdot u(X')^{-1} \cdot u_1(X').$$

By hypothesis $u(a') \neq 0$.

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Consider arbitrary equation $v_j(X) = 0$ of (3). We have to prove that $X = a$ is its solution.

Let

$$v_j = x_{1m}u_1 + \ldots + x_{rm}u_r + w,\ u_i \in \mathbb{Z}C,\ w \in TR + x_{r+1,m} \cdot \mathbb{Z}C + \ldots + x_{nm} \cdot \mathbb{Z}C.$$

As $v_j \in V$ and $\{v_1, \ldots, v_r\}$ is a basis of V, then

$$v_j(X) = v_1(X) \cdot u(X')^{-1} \cdot u_1(X') + \ldots + v_r(X) \cdot u(X')^{-1} \cdot u_1(X').$$

By hypothesis $u(a') \neq 0$.

Consider a specialization $X' \to a'$, it gives an epimorphism of rings $\mathbb{Z}C \to \mathbb{Z}A$. We prove (and it is very principal and hard step using the statement 2) for A) that this epimorphism possible to lift to an epimorphism $Q_0(C) \to Q(A)$, where $Q_0(C)$ is some subring of $Q(C)$ which contains $\mathbb{Z}C$ and the element $u(X')^{-1}$. Let β_i denote an image of $u(X')^{-1} \cdot u_i(X')$. Last epimorphism of rings defines an epimorphism of modules

$$\varphi : T \cdot Q_0(C) + x_{r+1,m} \cdot Q_0(C) + \ldots + x_{nm} \cdot Q_0(C) \to$$

$$\to T + x_{r+1,m} \cdot Q(A) + \ldots + x_{nm} \cdot Q(A).$$

We have

$$v_j(a', x_{1m}, \ldots, x_{nm}) = v_1(a', x_{1m}, \ldots, x_{nm})\beta_1 + \ldots + v_r(a', x_{1m}, \ldots, x_{nm})\beta_r.$$

Then apply to both parts an epimorphism of vector $Q(A)$-spases

$$T + x_{1m} \cdot Q(A) + \ldots + x_{nm} \cdot Q(A) \to T$$

which is defined by the mapping

$$x_{1m} \to a_{1m}, \ldots, x_{nm} \to a_{nm}$$

we have

$$v_j(a) = v_1(a)\beta_1 + \ldots + v_r(a)\beta_r = 0 \cdot \beta_1 + \ldots + 0 \cdot \beta_r = 0.$$

Now we can state that with condition $X' \in S$ the system (3) is equivalent to the finite subsystem $\Sigma_1 \cup \Sigma_2$. This is a contradiction with hypothesis. But we didn't tell about the proof of the statement 2) of the theorem1'.

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