Rigid Solvable Groups

Nikolay Romanovskiy

Institute of Mathematics, Novosibirsk, Russia

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1. Introduction

2. Algebraic geometry

3. EN groups
   - Which groups are equationally Noetherian?
   - Examples of groups which are not equationally Noetherian
   - Rigid groups
   - Separation, discrimination and universal theories

4. Divisible groups

5. Dimension theory

6. Irreducible sets

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10. Proof of the Theorem 1
Algebraic geometry over groups and other algebraic systems. G. Baumslag, O. Kharlampovich, A. Myasnikov, B. Plotkin, V. Remeslennikov.

Introduction to algebraic geometry over groups:

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Introduction to algebraic geometry over groups:

Our talk based on following papers:
N.S.Romanovskiy, *On representations of rigid soluble groups by defining relations*, submitted for publication.
$G \triangleright A$, $A$ is abelian. $G$ acts by conjugation: $a \rightarrow a^g = g^{-1}ag$.

$G/A$ acts, $A$ is a right $\mathbb{Z}[G/A]$-module.

$u = \alpha_1\bar{g}_1 + \ldots + \alpha_n\bar{g}_n \in \mathbb{Z}[G/A]$, $a^u = (a^{g_1})^{\alpha_1} \cdots (a^{g_n})^{\alpha_n}$.

**Definition**

$m$-rigid group $G$: there is a normal series

$$G = G_1 > G_2 > \ldots > G_m > G_{m+1} = 1,$$

$G_i/G_{i+1}$ are abelian and considering as right $\mathbb{Z}[G/G_i]$-modules have no torsion.

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1) Free solvable group is rigid, rigid series consists of commutator subgroups.

2) \( W = A_m \wr (A_{m-1} \wr \ldots \wr A_1) \ldots \), where \( A_i \) are free abelian groups.

Subgroups of rigid groups are rigid too: \( G \supset H \), \( H_i = H \cap G_i \). Corresponding series for \( H \) may be shorter.
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$G$-group, $G$-subgroups, $G$-homomorphism, ...

$F = G \ast \langle x_1, \ldots, x_n \rangle$, $x = (x_1, \ldots, x_n)$.

Equation $\nu(x) = 1$, $\nu(x) \in F$.

$F$ is a group of all equations.

$\{\nu_i(x) = 1 \ (i \in I)\}$, set of solutions $S \subseteq G^n$ is called algebraic set.

Annihilator of $S$: $I(S) = \{\nu(x) \in F \mid \nu(S) = 1\}$.

Coordinate group of $S$: $\Gamma(S) = F/I(S)$.

$\Gamma(S) \geq G$.

A category of algebraic sets is dual to a category of coordinate groups.
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Group of all equations $D = \langle G, x_1, \ldots, x_n \rangle$:
$x \rightarrow (a_1, \ldots, a_n) \in G^n$ possible to continue to $G$-epimorphism $D \rightarrow G$.

$D = F/H$. $H$ is maximal $= I(G^n)$ is a set of all $G$-identities, $F/H = \Gamma(G^n)$.
In particular, if $G \in \mathcal{M}$, then $\Gamma(G^n) \in \mathcal{M}$.
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The intersection of algebraic sets $\bigcap S_i$ is algebraic, but the union $S_1 \cup S_2$ is not in general case and very often when $G$ is solvable.

Zariski topology on $G^n$: take the algebraic sets as a sub-basis for the closed sets.

The topology is Noetherian, if every properly descending chain of closed subsets is finite. In this case $S = S_1 \cup \ldots \cup S_k$, $S_i \not\subseteq S_j$, $S_i$ are irreducible components.

We say that given group is equationally Noetherian (EN) if for any $n$ arbitrary system of equations on $x_1, \ldots, x_n$ over this group is equivalent to some finite subsystem.

**Proposition 1** $G$ is EN $\iff$ for any $n$ Zariski topology on $G^n$ is Noetherian. In this case irreducible sets are algebraic.

Hard to study algebraic geometry over given group without last property. So, to be equationally Noetherian group is necessary condition for good algebraic geometry.
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   - Examples of groups which are not equationally Noetherian
   - Rigid groups
   - Separation, discrimination and universal theories

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2) Linear groups, in particular, free groups.

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10. Proof of the Theorem 1
Example 1.

\[ A = \langle a_1, a_2, \ldots, b_1, b_2, \ldots | \forall_{2}, [b_1, a_1] = 1, [b_2, a_1] = [b_2, a_2] = 1, \ldots, [b_n, a_1] = \ldots = [b_n, a_n] = 1, \ldots \rangle \]

A system of equations \{[x, a_i] = 1\} isn’t equivalent to a finite subsystem.

Example 2.

\[ H = \langle c, d \rangle \] is a free centre-by-metabelian group, \([H, H]\) contains a free nilpotent group of class 2 with a countable basis \{a_1, a_2, \ldots, b_1, b_2, \ldots\}.

\[ G = \langle c, d \mid \ldots \rangle \geq A. \]

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Theorem 1 (R, 2009)

Rigid groups are equationally Noetherian.

For free solvable groups and f.g. rigid groups it was proved by GR in 2007.

Main purpose of algebraic geometry over group: to describe algebraic sets. For EN groups we have two problems.
1. To describe irreducible sets.
2. To define when the union $S_1 \cup \ldots \cup S_k$ of irreducible algebraic sets is algebraic.
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3 EN groups

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Let $H_1, H_2$ be $G$-groups.

$H_2$ separates $H_1$: $1 \neq a \in H_1$, $\varphi : H_1 \to H_2$, $a\varphi \neq 1$.

$H_2$ discriminates $H_1$:
1) $\{a_1, \ldots, a_n\}$, $1 \neq a_i \in H_1$, $\varphi : H_1 \to H_2$, $a_i\varphi \neq 1$,

or 2)$\{a_1, \ldots, a_n\}$, $a_i \neq a_j \in H_1$, $\varphi : H_1 \to H_2$, $a_i\varphi \neq a_j\varphi$.

**Proposition 2** Let $H = \langle G, y_1, \ldots, y_n \rangle$.
1) $H$ is a coordinate group of an algebraic set $S \subseteq G^n$ on $y_1, \ldots, y_n$ $\iff$ $H$ is $G$-separated by $G$.

2) If $G \in EN$ then $H$ is a coordinate group of an irreducible algebraic set $S \subseteq G^n$ on $y_1, \ldots, y_n$ $\iff$ $H$ is $G$-discriminated by $G$. 
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2) If $G \in EN$ then $H$ is a coordinate group of an irreducible algebraic set $S \subseteq G^n$ on $y_1, \ldots, y_n$ $\iff$ $H$ is $G$-discriminated by $G$. 
Let $H$ be $G$-group. Denote by $U_G(H)$ the universal theory ($\forall$-theory) of $H$ with constants from $G$.
\[
\forall x_1, \ldots, x_n \Phi(x), \quad \Phi(x) : \bigvee, \bigwedge \nu(x) = 1, \nu(x) \neq 1.
\]

**Proposition 3** Let $H_1, H_2$ be $G$-groups which are EN by equations with constants from $G$. Then $U_G(H_1) = U_G(H_2) \iff H_1$ is locally discriminated by $H_2$ and $H_2$ is locally discriminated by $H_1$.

**Proposition 4** Let $H = \langle G, y_1, \ldots, y_n \rangle$ be EN by equations with constants from $G$. Then $H$ is a coordinate group of an irreducible algebraic set $S \subseteq G^n$ on $y_1, \ldots, y_n \iff U_G(H) = U_G(G)$. 
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In the papers

V.N.Remeslennikov, N.S.Romanovskiy, Irreducible algebraic sets in metabelian groups, Algebra and Logic, 44(5), 2005, pp. 336-347,
N.S.Romanovskiy, Algebraic sets in metabelian groups, Algebra and Logic, 46(4), 2005, pp. 503-513
we described algebraic sets in the dimension 1 over free metabelian group.
This description doesn’t give any optimism that possible to get good information about all algebraic sets over arbitrary (finitely generated) rigid group.

To find such class of m-rigid groups, that any m-rigid group can be embedded into some group of this class and the algebraic geometry over groups of the class will be "good".
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\[ G_i/G_{i+1} \text{ are abelian and considering as right } \mathbb{Z}[G/G_i]\text{-modules have no torsion.} \]

We can describe rigid series.

\[ \delta_1 = x_1, \; \delta_2 = [x_1, x_2], \; \delta_3 = [[x_1, x_2], [x_3, x_4]], \ldots \]

define corresponding commutator subgroups.

Take \( a_m = \delta_m(\ldots) \neq 1, \; a_{m-1} = \delta_{m-1}(\ldots) \notin G_m, \; a_{m-2} = \delta_{m-1}(\ldots) \notin G_{m-1}, \ldots \)

\( G_m \) is a centralizer of \( a_m \), \( G_{m-1} \) is a centralizer of \( a_{m-1} \) modulo \( G_m \), \( G_{m-2} \) is a centralizer of \( a_{m-2} \) modulo \( G_{m-1} \), \ldots

\[ [a_m, x] = a_m^{-1}x^{-1}a_mx = a_m^{x^{-1}} = 1 \iff x \in G_m. \]
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$G_m$ is a centralizer of $a_m$, $G_{m-1}$ is a centralizer of $a_{m-1}$ modulo $G_m$, $G_{m-2}$ is a centralizer of $a_{m-2}$ modulo $G_{m-1}, \ldots$

$[a_m, x] = a_m^{-1}x^{-1}a_mx = a_m^{x^{-1}} = 1 \iff x \in G_m$. 
Some facts about rings.
Right Ore domain $R$: no zero divisors and for any $a, b \in R$ there is a nontrivial solution of equation $ax = by$.
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Definition

Rigid group $G$ is called divisible if any factor $T_i = G_i/G_{i+1}$ is a divisible module over the ring $\mathbb{Z}[G/G_i]$ or, in other words, $T_i$ is a vector space over skew field of fractions $Q(G/G_i)$.

Let $\alpha_1, \ldots, \alpha_m$ be nonzero cardinalities. Construct group $M(\alpha_1, \ldots, \alpha_m)$ by induction. $M(\alpha_1)$ is a direct sum of $\alpha_1$ copies of $Q$. $A = M(\alpha_1, \ldots, \alpha_{m-1})$. Let $T$ be a vector space with a basis of cardinality $\alpha_m$ over the skew field $Q(A)$. Then set

$$M(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$ 

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Theorem 2 (R, 2009)

*Arbitrary m-rigid group can be embedded into some decomposed divisible m-rigid group.*

Finitely generated rigid groups are exactly finitely generated subgroups of iterated wreath products of free abelian groups $W = A_n \wr (A_{n-1} \wr (\ldots \wr A_1)\ldots)$.

$G \supseteq H, \ H_i = H \cap G_i, \ H_i/H_{i+1} \leq G_i/G_{i+1}, \ \mathbb{Z}[H/H_i] \leq \mathbb{Z}[G/G_i]$. We say that $H$ is embedded into $G$ with preserving linear independence, if any system elements of $H_i/H_{i+1}$ linear independent over the ring $\mathbb{Z}[H/H_i]$ has to be linear independent over the ring $\mathbb{Z}[G/G_i]$. 
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Theorem 3 (R, 2008)

Let $G$ be a $m$-rigid subgroup of divisible rigid group $D$. Then there is a minimal divisible subgroup containing $G$, let it be $G = \text{divisible closure of } G$ in $D$. This subgroup $G$ is $m$-rigid and $G_i / G_{i+1}$ is generated by the set $G_i / G_{i+1}$ as a vector space over $Q(G / G_i)$.

Natural question: Let $G_1$ and $G_2$ be two divisible closures of $G$, are they $G$-isomorphic?

NO, in general case, but YES with adding condition:

Theorem 4 (R, 2008)

For given $m$-rigid group $G$ there is such divisible closure $\hat{G}$ that $G$ is embedded into $\hat{G}$ with preserving linear independence. We call $\hat{G}$ divisible completion of $G$. Any two divisible completions of $G$ are $G$-isomorphic.
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\( T_i = G_i/G_{i+1} \) is a torsion free module over the ring \( \mathbb{Z}[G/G_i] \).

\( \tau_i(G) = \text{rank } T_i, \ r(G) = (\tau_1(G), \ldots, \tau_m(G)). \)

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\( T_i \) embeds into the right vector space \( V_i = T_i \otimes_{\mathbb{Z}[G/G_i]} Q(G/G_i), \ \tau_i(G) = \text{dim } V_i. \)
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Let $G$ be $m$-rigid group, $S \subseteq G^n$ be an irreducible algebraic set, $\Gamma = \Gamma(S)$.

1) Then $\Gamma$ is $m$-rigid.

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The tuple $d(S) = (d_1(S), \ldots, d_p(S))$ is called a dimension of irreducible algebraic set $S$. For finitely generated $G$ all ranks $r_i(G)$ and $r_i(\Gamma)$ are finite and so $d(S) = r(\Gamma) - r(G)$. 
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Let $G$ be a rigid group, $S$ and $P$ irreducible algebraic subsets of $G^n$. If $S \supset P$, then $d(S) > d(P)$ in lexicographic order.

Remind that topological dimension of given topological space by definition is equal to supremum of lengths of chains $S_1 > S_2 > \ldots > S_m$ irreducible subsets.

Corollary

If $G$ be a $m$-rigid group then the topological dimension of the space $G^n$ is finite and doesn’t exceed the number $(n + 1)^m$.

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Let \( M = M(\alpha_1, \ldots, \alpha_m) \). Then finitely generated \( M \)-group \( G \) is a coordinate group of some irreducible algebraic set over \( M \) if and only if \( G \) is \( m \)-rigid and \( M \) is embedded into \( G \) with preserving linear independence.

We deduce the theorem 8 from following statement.

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Let a group \( G \) contain \( M = M(\alpha_1, \ldots, \alpha_m) \) as a subgroup. Then \( G \) is \( M \)-universally equivalent to \( M \) if and only if \( G \) is \( m \)-rigid and \( M \) is embedded into \( G \) with preserving linear independence.
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Malcev proved that free solvable group of class $\geq 2$ has undecidable elementary theory. The universal theory of free metabelian group was studied by Chapuis, Remeslennikov and Stohr, in particular, it is decidable. Free solvable groups of given class $\geq 3$ and different ranks are universally equivalent too and Chapuis proved that their universal theory is undecidable if the universal theory of rational numbers is undecidable (10-th Hilbert problem for rational numbers). Nevertheless, Chapuis proved that the the universal theory of an iterated wreath product of of free abelian groups is decidable.
We construct a recursive system of $\forall$-axioms which define $m$-rigid groups in the class of all $m$-soluble groups.

Let $F$ denote a free solvable group of class $m$, $G$ denote an arbitrary $m$-rigid group, $W$ denote the iterated wreath product of $m$ cyclic groups. For corresponding universal theories we prove conclusions

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The universal theories of \( W \) and the group \( M = M(\alpha_1, \ldots, \alpha_m) \) coincide.
We prove that the universal theory of \( W \) with constants from \( W \) is undecidable.
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**Definition**

*m-graduated rigid group* \( G \) with graduation \( \varepsilon \): there is a normal series

\[ G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_m \supseteq G_{m+1} = 1, \]

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Abelian torsion free group may have different \( m \)-graduations:
\((1,0,\ldots,0,0), \ldots, (0,0,\ldots,0,1)\).
\( m \)-rigid group has only one \( m \)-graduation: \((1,1,\ldots,1,1)\).

If \( H \leq G \), \( H_i = H \cap G_i \), \( \varepsilon' \) is a corresponding graduation for \( H \), then \( \varepsilon'_i \leq \varepsilon_i \).

A homomorphism of \( m \)-graduated rigid groups:
\( \varphi : G \rightarrow H, \ G_i \varphi \leq H_i \).
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We have a category of \( m \)-graduated rigid groups. Next theorem actually states that there is a coproduct in this category.
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We have a category of $m$-graduated rigid groups. Next theorem actually states that there is a coproduct in this category.
**Theorem 11 (R, 2010).** Let $G$ and $H$ be two $m$-graduated rigid groups. Then there is $m$-graduated rigid group $G \circ H$, which is called $m$-rigid product of $G$ and $H$, with following properties.

1) Groups $G$ and $H$ are subgroups of $G \circ H$ and generate it.
2) Any homomorphisms

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As coproduct the operation $\circ$ is defined uniquely, commutative and associative.
Let $F_1, \ldots, F_n$ be infinite cyclic groups with $m$-graduation $(1, 0, \ldots, 0)$. Then their $m$-rigid product $F_1 \circ \ldots \circ F_n$ is a free $m$-solvable group of the rank $n$.

Let $A$ and $B$ be rigid groups with $m$-graduations $(0, \ldots, 0, 1)$, and $(1, \ldots, 1, 0)$. Then the product $A \circ B$ is isomorphic to $A \wr B$.

**Theorem 12 (R, 2010).** Let $G$ be $m$-rigid group and $F$ be free $m$-solvable group (for rank 1 with $m$-graduation $(1, 0, \ldots, 0)$). Then $m$-rigid product $G \circ F$ is $G$-discriminated by $G$.

**Corollary.** Let $\{x_1, \ldots, x_n\}$ be a basis of the group $F$ in the theorem. Then $G \circ F$ is a coordinate group of the affine space $G^n$ on $x_1, \ldots, x_n$ and this space is irreducible.
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**Theorem 13 (R)**

*For $m$-rigid $n$-generated groups the lengths of strongly ascending (descending) chains of ideals are bounded by some function of $m$ and $n.*
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**Theorem 13 (R)**

*For m-rigid n-generated groups the lengths of strongly ascending (descending) chains of ideals are bounded by some function of $m$ and $n$.***
$\Sigma_m = \text{all } \leq m\text{-rigid groups.}$ Arbitrary $n$-generated group of $\Sigma_m$ is a factor group of a free solvable group $F_{m,n}$ of length $m$ with a basis $\{x_1, \ldots, x_n\}$ by some ideal. How can we represent groups in $\Sigma_m$ by defining relations? $R = R(x_1, \ldots, x_n)$ — some set of group words on $x_1, \ldots, x_n$. In a classic case the group $\langle x_1, \ldots, x_n \mid R \rangle$ is a factor group of a free group by the least normal subgroup containing $R$. In our case not always there is a least ideal of $F_{m,n}$ containing $R$.

Example: $m = 2, \ n = 3, \ R = \{[x_1, x_2]^{x_3-1}\}$. If $G \in \Sigma_2$ is generated by $x_1, x_2, x_3$, and $[x_1, x_2]^{x_3-1} = 1$ then or $[x_1, x_2] = 1$, or $x_3 \in G_2$.

First group is defined in the variety $\mathfrak{A}^2$ by generators $x_1, x_2, x_3$ and relation $[x_1, x_2] = 1$, here $x_3 \not\in G_2$. Second group is defined by relations $[x_3, F'_{2,3}] = 1$, here $[x_1, x_2] \neq 1, \ x_3 \in G_2$. And there isn’t a group of $\Sigma_2$ with relation $[x_1, x_2]^{x_3-1} = 1$ which covers both groups.
\[ \Sigma_m(R) = \text{all groups of } \Sigma_m \text{ generated by } x_1, \ldots, x_n \text{ with relations } R. \] Maximal group = there is no proper covering in \( \Sigma_m(R) \).

Theorem 13 \( \Rightarrow \) for any group of \( \Sigma_m(R) \) there is a maximal covering. Any maximal groups of \( \Sigma_m(R) \) possible to understand as a group defining in \( \Sigma_m \) by generators \( x_1, \ldots, x_n \) and relations \( R \).

**Theorem 14 (R)**

For arbitrary \( R \) the set \( \Sigma_m(R) \) contains only finite number of maximal groups.

\( R \) is called complete set of defining relations if \( \Sigma_m(R) \) contains only single maximal group, so this group is defined by relations \( R \) uniquely.

**Theorem 15 (R)**

Arbitrary finitely generated group of \( \Sigma_m \) is finitely completely presented, it means that there is a finite complete set of defining relations.
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**Theorem 16 (R)**

*For arbitrary finite set of relations $R = R(x_1, \ldots, x_n)$ there is an effectively procedure of constructing of some finite set $\Omega_m(R)$ of canonical representations on generators $x_1, \ldots, x_n$ of groups of $\Sigma_m(R)$ such that $\Omega_m(R)$ contains all maximal groups of $\Sigma_m(R)$.*

Note, that we can’t effectively define: which groups of $\Omega_m(R)$ are exactly maximal. But in any case we can define: is given word $\nu(x_1, \ldots, x_n)$ an implication of relations $R$ in $\Sigma_m$ or not?
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Theorem 1 (R, 2009)

Rigid groups are equationally Noetherian.

The group $M(\alpha_1, \ldots, \alpha_m)$ is constructed by induction. $M(\alpha_1)$ is a direct sum of $\alpha_1$ copies of $\mathbb{Q}$. $A = M(\alpha_1, \ldots, \alpha_{m-1})$. Let $T$ be a vector space with a basis of cardinality $\alpha_m$ over the skew field $\mathbb{Q}(A)$. Then set

$$M = M(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

$M$ is a semidirect product $A_1A_2\ldots A_m$ of abelian groups $A_i$, $A_m = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$. $M_i = A_iA_{i+1}\ldots A_m$ are members of rigid series.
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Let $1 \neq a_i \in A_i$. Note that $A_i$ is exactly the centralizer of $a_i$ in $M$, so $A_i = \{x \in M \mid [a_i, x] = 1\}$.

Consider following set of variables 
$X = \{x_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, m\}$ and its subset 
$X' = \{x_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, m - 1\}$. We suppose that 
values of the variable $x_{ij}$ belong to $A_j$. Call $x_{ij}$ special variables. 
Possible to consider them as usual variables with adding conditions 
$[x_{ij}, a_j] = 1$. Last equations define an algebraic set which we denote 
by $\Omega$. We define also special algebraic equations and special 
algebraic sets (subsets of $\Omega$). Also $\Omega$ possible to identify with $M^n$.

Let $x_1 = x_{11}x_{12}\ldots x_{1m}, \ldots, x_n = x_{n1}x_{n2}\ldots x_{nm}$. We see that any 
elements of $M$ can be values of $x_i$. So usual equations on $x_1, \ldots, x_n$ 
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Theorem 1’ (R, 2009)

1) The group $M$ is EN by special equations on $X$.
2) Let $S$ be a special irreducible algebraic subset of $M^n$ and $\varphi : \Gamma(S) \to M$ be arbitrary specialization. Then $\ker \varphi$ is separated by nilpotent torsion free groups.

For our purpose we need only statement 1), but the proof by induction will use statement 2). By induction suppose that for $A = M(\alpha_1, \ldots, \alpha_{m-1})$ corresponding statements hold.
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For our purpose we need only statement 1), but the proof by induction will use statement 2).
By induction suppose that for $A = M(\alpha_1, \ldots, \alpha_{m-1})$ corresponding statements hold.
Now we construct a group of special equations $M[X]$. Let the group $A_1[x_{11}, \ldots, x_{n1}]$ be equal a direct product of $A_1$ and a free abelian group with a basis $\{x_{11}, \ldots, x_{n1}\}$. Suppose by induction the group $B = A[X']$ is constructed and let it be generated by its subgroup $A$ and the set $X'$ and there be a decomposion of $B$ as semidirect product $B_1 B_2 \ldots B_{m-1}$, where the abelian group $B_j$ contains $A_j$ and $x_{ij} \in B_j$. Suppose that any mapping $x_{ij} \to a_{ij} \in A_j$ ($i = 1, \ldots, n; j = 1, \ldots, m - 1$) possible to continue to an $A$-epimorphism $B \to A$. 
Consider a direct sum of the module $T \otimes_{\mathbb{Z}A} \mathbb{Z}B$ and the right free $\mathbb{Z}B$-module with the basis \{${x_1m, \ldots, x_nm}$\}. Set

$$M[X] = \left( \begin{array}{cc} B & 0 \\ T \otimes_{\mathbb{Z}A} \mathbb{Z}B + x_1m \cdot \mathbb{Z}B + \ldots + x_nm \cdot \mathbb{Z}B & 1 \end{array} \right),$$

here we indentify the element $x_{ij}$ for $j < m$ with the matrix

$$\begin{pmatrix} x_{ij} & 0 \\ 0 & 1 \end{pmatrix},$$

and the element $x_{im}$ with the matrix

$$\begin{pmatrix} 1 & 0 \\ x_{im} & 1 \end{pmatrix}.$$  The group $M[X]$ is generated by its subgroup $M$ and the set $X$ and $M[X] = B \cdot B_m$, where the subgroup $B_m$ is isomorphic to the additive group of the module

$$T \otimes_{\mathbb{Z}A} \mathbb{Z}B + x_1m \cdot \mathbb{Z}B + \ldots + x_nm \cdot \mathbb{Z}B.$$

We prove that any mapping $x_{ij} \rightarrow a_{ij} \in A_j$ possible to continue to an $M$-epimorphism $M[X] \rightarrow M$. It means that $M[X]$ is a group of special equations on $X$ over $M$. 
An equation of the type $u(X') = 0$, where $u(X') \in \mathbb{Z}B$, is called a group ring equation, for this equation we find solutions with restriction $x_{ij} \in A_j$. It is important here: arbitrary group ring equation is equivalent to some disjunction of finite number of finite systems of group equations.

For example the equation $V_1 + V_2 + V_3 - V_4 - 2V_5 = 0$, where $V_i \in A[X']$, is equivalent to
\[
((V_1 = V_4) \land (V_2 = V_3 = V_5)) \lor ((V_2 = V_4) \land (V_1 = V_3 = V_5))\lor \\
\lor ((V_3 = V_4) \land (V_1 = V_2 = V_5)).
\]

So, a group ring equation defines a closed subset in $A^n$. Easy to note that any closed subset of $A^n$ can be defined by some group ring equation.
Possible to realize arbitrary group ring equation \( u(X') = 0 \) as a
group equation \( v(X) = 1, \ v(X) \in M[X] : \) take some nontrivial
element \( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in M \) and set \( v(X) = \begin{pmatrix} 1 & 0 \\ t \cdot u(X') & 1 \end{pmatrix}. \)

We prove also that if \( L \) is an irreducible subset of \( M^n \) then the
closure of its projection on \( A^n \) is irreducible too.

Now come immediately to the proof of the theorem 1'.

Consider an arbitrary system of special equations on \( X \) over \( M : \)

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\{ w_j(X) = 1, \ j \in J. \quad (1) \]

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By induction the system (2) is equivalent to some its finite subsystem. If (2) has no solutions then some finite subsystem of (1) has no solutions. So, suppose that the system (2) defines nonempty closed subset in \( A^n \).

Suppose we have a counterexample to the statement 1) of our theorem. Then there is nonempty closed subset \( S \) of \( A^n \) such that the system (3) with condition \( X' \in S \) isn’t equivalent to finite subsystem. We can suppose that \( S \) is minimal with this property, then it is irreducible.
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So, for any closed proper nonempty subset $P \subset S$ the system (3) with condition $X' \in P$ is equivalent to some its finite subsystem, but with condition $X' \in S$ it isn’t equivalent to finite subsystem.

Let $C$ denote the coordinate group $\Gamma(S)$. This group is generated by $A$ and the images of the elements $x_{ij} \in X'$ which we will denote by the same symbols. As this group is discriminated by $A$ then it is soluble torsion free. Consider the skew field of fractions $Q(C)$ and its subring $R$ generated by $Q(A)$ and $X'$. We remember that $T$ is a right vector space over $Q(A)$. Embed it into the free $R$-module $TR$ with the same basis. Consider a direct sum of $TR$ and a free right $\mathbb{Z}C$-module with the basis $\{x_{1m}, \ldots, x_{nm}\}$. Then take a group

$$D = \begin{pmatrix} C \\ TR + x_{1m} \cdot \mathbb{Z}C + \cdots + x_{nm} \cdot \mathbb{Z}C & 0 \end{pmatrix}$$

and identify $x_{ij}$ for $j < m$ with \( \begin{pmatrix} x_{ij} & 0 \\ 0 & 1 \end{pmatrix} \), and $x_{im}$ with \( \begin{pmatrix} 1 & 0 \\ x_{im} & 1 \end{pmatrix} \).
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We prove that the group $D$ is a group of special equations on $X$ over $M$ with condition $X' \in S$.

Therefore possible to suppose that the left parts of equations (3) belong to the module $TR + x_{1m} \cdot \mathbb{Z}C + \ldots + x_{nm} \cdot \mathbb{Z}C$.

Embed this module to the right vector space $T \cdot Q(C) + x_{1m} \cdot Q(C) + \ldots + x_{nm} \cdot Q(C)$ and denote by $V$ a subspace generated by $v_j \ (j \in J)$.

We prove that $V \cap T \cdot Q(C) = 0$, therefore the projection $V$ onto the space $x_{1m} \cdot Q(C) + \ldots + x_{nm} \cdot Q(C)$ is injective. Then $\dim V \leq n$. Let $\{v_1, \ldots, v_r\}$ be a basis of $V$. By elementary transformations over the ring $\mathbb{Z}C$ we make this system generalized diagonal with the same element $u$ in diagonal.
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We prove that the group $D$ is a group of special equations on $X$ over $M$ with condition $X' \in S$.

Therefore possible to suppose that the left parts of equations (3) belong to the module $TR + x_1m \cdot \mathbb{Z}C + \ldots + x_nm \cdot \mathbb{Z}C$.

Embed this module to the right vector space $T \cdot Q(C) + x_1m \cdot Q(C) + \ldots + x_nm \cdot Q(C)$ and denote by $V$ a subspace generated by $v_j$ ($j \in J$).

We prove that $V \cap T \cdot Q(C) = 0$, therefore the projection $V$ onto the space $x_1m \cdot Q(C) + \ldots + x_nm \cdot Q(C)$ is injective. Then $\dim V \leq n$. Let $\{v_1, \ldots, v_r\}$ be a basis of $V$. By elementary transformations over the ring $\mathbb{Z}C$ we make this system generalized diagonal with the same element $u$ in diagonal.
Possible suppose that

\[ v_1 = x_1 m u + v_1', \ldots, v_r = x_r m u + v_r', \quad v_i' \in TR + x_{r+1,m} \mathbb{Z} C + \ldots + x_{nm} \mathbb{Z} C. \]

Consider the ring equation \( u(X') = 0 \) with condition \( X' \in S \). It defines proper (may be empty) closed subset of \( S \), let it be \( P \). If \( P \) isn’t empty there is a finite subsystem \( \Sigma_1 \) of the system (3) which is equivalent to (3) with condition \( X' \in P \). For empty \( P \) suppose \( \Sigma_1 \) is empty.

Denote by \( \Sigma_2 \) the system \( v_1(X) = 0, \ldots, v_r(X) = 0 \).

We will prove that with condition \( X' \in S \setminus P \) the system (3) is equivalent to \( \Sigma_2 \).
Possible suppose that

\[ v_1 = x_1 m u + v'_1, \ldots, v_r = x_r m u + v'_r, \quad v'_i \in TR + x_{r+1,m} \mathbb{Z} C + \ldots + x_{nm} \mathbb{Z} C. \]

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We will prove that with condition \( X' \in S \setminus P \) the system (3) is equivalent to \( \Sigma_2 \).
Let
\[ a = (a_{ij} \in A_{ij} \mid i = 1, \ldots, n; \ j = 1, \ldots, m) \]
be a solution of the system \( \Sigma_2 \) and
\[ a' = (a_{ij} \mid i = 1, \ldots, n; \ j = 1, \ldots, m - 1) \in S \setminus P. \]

Consider arbitrary equation \( v_j(X) = 0 \) of (3). We have to prove that \( X = a \) is its solution.

Let
\[ v_j = x_{1m}u_1 + \ldots + x_{rm}u_r + w, \ u_i \in \mathbb{Z}C, \ w \in TR + x_{r+1,m} \mathbb{Z}C + \ldots + x_{nm} \mathbb{Z}C. \]

As \( v_j \in V \) and \( \{v_1, \ldots, v_r\} \) is a basis of \( V \), then
\[ v_j(X) = v_1(X) \cdot u(X')^{-1} \cdot u_1(X') + \ldots + v_r(X) \cdot u(X')^{-1} \cdot u_1(X'). \]

By hypothesis \( u(a') \neq 0 \).
Let
\[ a = (a_{ij} \in A_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, m) \]
be a solution of the system $\Sigma_2$ and
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As $v_j \in V$ and $\{v_1, \ldots, v_r\}$ is a basis of $V$, then
\[ v_j(X) = v_1(X) \cdot u(X')^{-1} \cdot u_1(X') + \ldots + v_r(X) \cdot u(X')^{-1} \cdot u_1(X'). \]

By hypothesis $u(a') \neq 0$. 
Consider a specialization $X' \rightarrow a'$, it gives an epimorphism of rings $\mathbb{Z}C \rightarrow \mathbb{Z}A$. We prove (and it is very principal and hard step using the statement 2) for $A$) that this epimorphism possible to lift to an epimorphism $Q_0(C) \rightarrow Q(A)$, where $Q_0(C)$ is some subring of $Q(C)$ which contains $\mathbb{Z}C$ and the element $u(X')^{-1}$. Let $\beta_i$ denote an image of $u(X')^{-1} \cdot u_i(X')$. Last epimorphism of rings defines an epimorphism of modules

$$
\varphi : T \cdot Q_0(C) + x_{r+1,m} \cdot Q_0(C) + \ldots + x_{nm} \cdot Q_0(C) \rightarrow
$$

$$
\rightarrow T + x_{r+1,m} \cdot Q(A) + \ldots + x_{nm} \cdot Q(A).
$$

We have

$$
v_j(a', x_1 m, \ldots, x_{nm}) = v_1(a', x_1 m, \ldots, x_{nm})\beta_1 + \ldots + v_r(a', x_1 m, \ldots, x_{nm})\beta_r.
$$
Then apply to both parts an epimorphism of vector $Q(A)$-spases

$$T + x_1m \cdot Q(A) + \ldots + x_{nm} \cdot Q(A) \rightarrow T$$

which is defined by the mapping

$$x_1m \rightarrow a_1m, \ldots, x_{nm} \rightarrow a_{nm}$$

we have

$$v_j(a) = v_1(a)\beta_1 + \ldots + v_r(a)\beta_r = 0 \cdot \beta_1 + \ldots + 0 \cdot \beta_r = 0.$$ 

Now we can state that with condition $X' \in S$ the system (3) is equivalent to the finite subsystem $\Sigma_1 \cup \Sigma_2$. This is a contradiction with hypothesis. But we didn’t tell about the proof of the statement 2) of the theorem1’.
Then apply to both parts an epimorphism of vector $Q(A)$-spases

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