Generic reducibilities

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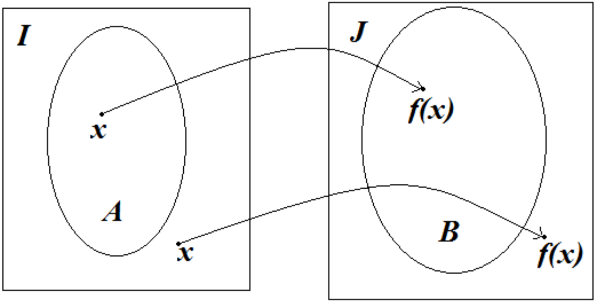
Classical reducibility



Definition

$A \subseteq I$ reduces to $B \subseteq J$, if there is a computable function $f : I \to J$ such that $x \in A \Leftrightarrow f(x) \in B$ for all $x \in I$.

Properties of the classical reducibility

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For all sets $A, B, C \subseteq I$

1. $A \leq A$ – reflexivity,
2. $A \leq B$ and $B \leq C$ implies $A \leq C$ – transitivity,
3. if $A \leq B$ and B is decidable, then A is decidable – preserving of decidability.

Generic reducibility

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Set $A \subseteq I$ *generically reduces to* $B \subseteq J$ ($A \leq_{Gen} B$) if there is a computable function $f : I \times \mathbb{N} \to J \cup \{?\}$ such that

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4. $\forall x \in I$ if $f(x,0) \neq ?$, then
   - $x \in A \Rightarrow \forall n \in \mathbb{N}$ $f(x,n) \in B$.
   - $x \notin A \Rightarrow \forall n \in \mathbb{N}$ $f(x,n) \notin B$.

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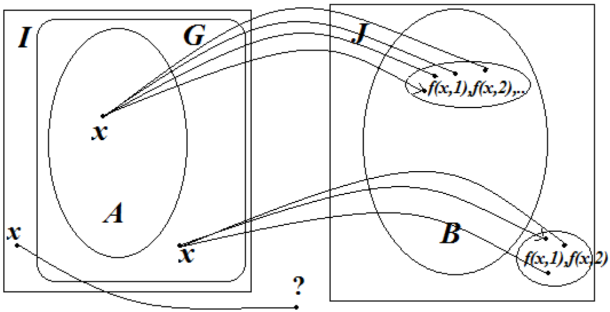

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- Computes $f(x,0)$. If $f(x,0) =?$ output ?
- If $f(x,0) \neq ?$, computes $f(x,1), f(x,2), \ldots$ until generic algorithm for B outputs on some $f(x,k)$ defined answer.

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- It will be correct answer for A.

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- Compute $f(x, 0)$. If $f(x, 0) = ?$, output $h(x, n) = ?$ for all n.
- If $f(x, 0) \neq ?$ computes $f(x, 1), f(x, 2), \ldots$ until $g(f(x, k), 0) \neq ?$.
- Now $h(x, n) = g(f(x, k), n)$ for all n.

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Definition

Set $B \subseteq \mathbb{N}$ is *simple* if B is c.e. and $\mathbb{N} \setminus B$ is immune.

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- $\mathbb{N} \setminus W$ is not negligible, so there is $x \notin W$ such that $f(x, 0) \neq ?$ and for all $f(x, k) \notin W$.

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If $W \subseteq \mathbb{N}$ is simple and non-generic, then $W \not\leq_{Gen} W$.

- Suppose $W \leq_{Gen} W$ by some generic reduction f.
- $\mathbb{N} \setminus W$ is not negligible, so there is $x \notin W$ such that $f(x,0) \neq ?$ and for all $f(x,k) \notin W$.
- $\{f(x,0), f(x,1), \dots, \}$ is infinite c.e. subset of immune set $\mathbb{N} \setminus W$.

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- $W \cap S_i \neq \varnothing$ for every infinite c.e. S_i.
- For every n there is only one element of size n in W, so

$$\rho(W) = \lim_{n\to\infty} \frac{|\mathbb{N}_n \cap W|}{|\mathbb{N}_n|} = \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

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Theorem

- W is super undecidable.
- There is no effective non-negligible cloning $C : \mathbb{N} \to P(\mathbb{N})$ such $W = C(S)$ for some c.e. set $S \subseteq \mathbb{N}$.

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- Now define $f(x, k) = M_k$.