Amplification of algorithmic problems: cryptographic problems

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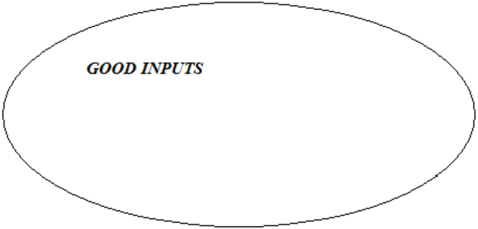
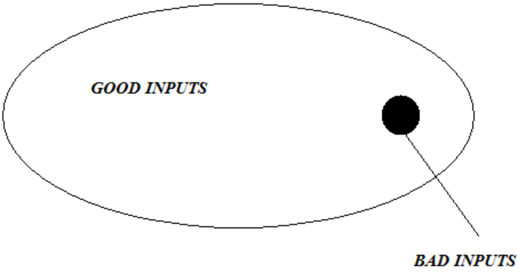
Classical approach



Algorithm works correctly on **all** inputs.

Generic approach



Algorithm works correctly on **almost all** inputs but can ignore some.

Asymptotic density

Definition

Let I be all inputs, I_n – all inputs of size n. **Asymptotic density** of set $S \subseteq I$

$$\mu(S) = \lim_{n \to \infty} \frac{|S \cap I_n|}{|I_n|}.$$

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$$\mu(S) = \lim_{n \to \infty} \frac{|S \cap I_n|}{|I_n|}.$$

Remark

$\frac{|S \cap I_n|}{|I_n|}$ is the probability to get an input from S during random and uniform generation of inputs of size n.

Generic algorithms

Definition

Algorithm $\mathcal{A} : I \to J \cup \{?\}$ is called **generic**, if

1. for every $x \in I$ $\mathcal{A}(x)$ halts,
2. $\mu(\{x : \mathcal{A}(x) = ?\}) = 0$.

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Definition

Generic algorithm $\mathcal{A}$ computes a function $f : I \to J$, if for every $x \in I$

$$\mathcal{A}(x) \neq ? \Rightarrow f(x) = \mathcal{A}(x).$$

Cryptographic problems

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Cryptographic hypothesis

For problems 1-3 there are no polynomial (probabilistic) algorithms, deciding their for all inputs.

Generic amplification

Theorem

If there exists a polynomial generic algorithm for a cryptographic problem, then there exists a polynomial probabilistic algorithm deciding it for all inputs.

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Corollary

If a cryptographic problem is hard in worst case, then it hard in generic case.

Discrete logarithm problem

Problem

$I = \{(a, g, p),\ p$ – prime, g – primitive $GF(p)$, $a \in GF(p)^*\}$,
Compute the function $dl : I \to \mathbb{N}$, defined in the following way:

$$dl(a, g, p) = x \Leftrightarrow g^x = a \text{ в } GF(p).$$

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Applications in cryptography

Diffie-Hellman protocol, El-Gamal scheme, Massey-Omura cryptosystem.

Stratification

Definition

Sequence

$$\pi = \{(p_m, g_m),\ \ m \in \mathbb{N}, \text{where } p_m - \text{prime},$$

$$g_m - \text{primitive in } \ GF(p_m)\}$$

is called **exponential**, if $2^m < p_m < 2^{m+1}$ for all m.

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Stratificated Discrete logarithm problem

$I_\pi = \{(a, g, p),\ (p, g) \in \pi,\ a \in GF(p)^*\}$,
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Size of input (a, g, p) is the length of binary expansion of p. Input set of size n: all triples of type (a, g, p), where g, p are fixed, $a \in GF(p)^*$.

Discrete logarithm problem is generically hard

Theorem

If there is no polynomial probabilistic algorithm for dl, then there exists an exponential sequence π such that there is no probabilistic algorithm for dl_π.

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If there is no polynomial probabilistic algorithm for dl, then there exists an exponential sequence π such that there is no probabilistic algorithm for dl_π.

Theorem

If there is a polynomial generic algorithm for dl_π, then there is polynomial probabilistic algorithm computing dl_π for all inputs.

Amplification of Discrete logarithm

$(a, g, p) \to (ag^k, g, p)$, where k is randomly and uniformly generated number from 0 to $p - 2$.

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$ag^k = g^x \Rightarrow a = g^{x-k}$.

Problem of computing of square roots in groups of residue classes

Problem

$I = \{(a, m) : \ m = pq, p, q - \text{ primes}, \ a \in \mathbb{Z}/(m)\}$,
Compute the function $sr : I \to \mathbb{N}$, defined in the following way:

$$sr(a, m) = \begin{cases} x, & \text{if } x^2 = a \text{ в } \mathbb{Z}/(m), \\ 0, & \text{otherwise.} \end{cases}$$

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Applications in cryptography

Rabin cryptosystem.

Stratification

Definition

Sequence

$$\mu = \{m_k,\ k \in \mathbb{N}, \text{где } m_k - \text{произведение двух простых}\}$$

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$I_\mu = \{(a, m) :\ m \in \mu,\ a \in \mathbb{Z}/(m)\}$,
Compute the function $sr_\mu = sr|_{I_\mu}$.

Size of input (a, m) is the length of binary expansion of m. Set of inputs of size n: all pairs of type (a, m), where m is fixed, $a \in \mathbb{Z}/(m)$.

Square root problem is generically hard

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If there is no polynomial probabilistic algorithm for sr, then there exists an exponential sequence μ such that there is no probabilistic algorithm for sr_μ.

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If there is no polynomial probabilistic algorithm for sr, then there exists an exponential sequence μ such that there is no probabilistic algorithm for sr_{μ}.

Theorem

If there is a polynomial generic algorithm for sr_{μ}, then there is polynomial probabilistic algorithm computing sr_{μ} for all inputs.

Amplification of the square root problem

$(a, m) \to (ab^2, m)$, where b is randomly and uniformly generated element from $\mathbb{Z}/(m)$.

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$ab^2 = x^2 \Rightarrow a = (xb^{-1})^2$.

Searching graph isomorphism problem

Problem

$I = \{(G_1, G_2),\ G_1, G_2\ -\ \text{isomorphic graphs}\}$. Compute the function $sgi : I \to S_1 \cup S_2 \cup \ldots S_n \cup \ldots$, defined in the following way:

$$sgi(G_1, G_2) = \text{permutation of vertices – isomorphism } G_1 \text{ and } G_2.$$

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Applications in cryptography

Zero-knowledge proof (Shafi Goldwasser, Silvio Micali, and Charles Rackoff).

Denotation

Sequence of graphs

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Problem

$I_\gamma = \{(G_1, G_2) :\ G_2 \in \gamma\}$. Compute the function $sgi_\gamma = sgi|_{I_\gamma}$.

Size of input (G_1, G_2) is number of vertices of G_2. Set of inputs of size n: all pairs of type (G_1, G_2), where G_2 is fixed, G_1 is graph isomorphic to G_2.

Searching graph isomorphism problem is generically hard

Theorem

If there is no polynomial probabilistic algorithm for *sgi*, then there exists a graph sequence γ such that there is no probabilistic algorithm for sgi_{γ}.

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If there is no polynomial probabilistic algorithm for *sgi*, then there exists a graph sequence γ such that there is no probabilistic algorithm for sgi_γ.

Theorem

If there is a polynomial generic algorithm for sgi_γ, then there is polynomial probabilistic algorithm computing sgi_γ for all inputs.

Amplification of the Searching graph isomorphism problem

$(G_1, G_2) \to (\pi(G_1), G_2)$, where π is randomly and uniformly generated permutation from S_n.

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$\tau(\pi(G_1)) = G_2 \Rightarrow \chi = \tau\pi$.