Amplification of algorithmic problems: undecidable problems

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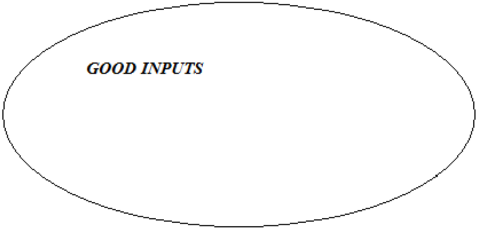
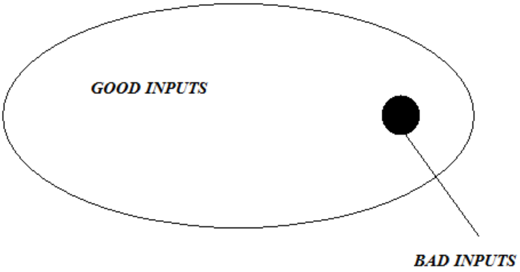
Classical approach



Algorithm works correctly on **all** inputs.

Generic approach



Algorithm works correctly on **almost all** inputs but can ignore some.

Asymptotic density

Definition

Let I be all inputs, I_n – all inputs of size n. **Asymptotic density** of set $S \subseteq I$

$$\mu(S) = \lim_{n \to \infty} \frac{|S \cap I_n|}{|I_n|}.$$

Asymptotic density

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Remark

$\frac{|S \cap I_n|}{|I_n|}$ is the probability to get an input from S during random and uniform generation of inputs of size n.

Generic sets

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- **generic** if $\mu(S) = 1$,
- **negligible** if $\mu(S) = 0$.

Generic decidability and undecidability

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Definition

Set $S \subseteq I$ is called **super undecidable** if it is not generically decidable.

Definition

Let I and J be input sets. Function $C : I \to P(J)$ is called **cloning** I to J if

1. $\forall x, y \in I \; x \neq y \to C(x) \cap C(y) = \emptyset$
2. There is an algorithm $E : I \times \mathbb{N} \to J$ such that for all $x \in I$

$$C(x) = \{E(x, 0), E(x, 1), \dots, \}$$

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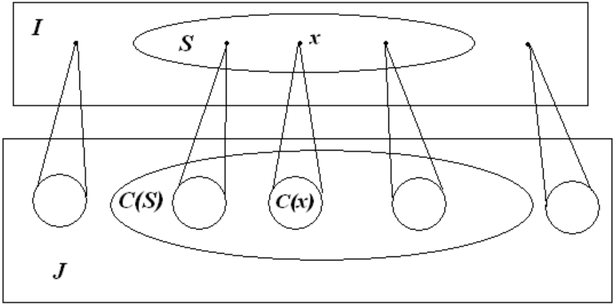
Definition

Cloning C is **non-negligible** if $\mu(C(x)) > 0$ for all $x \in I$.

Generic amplification

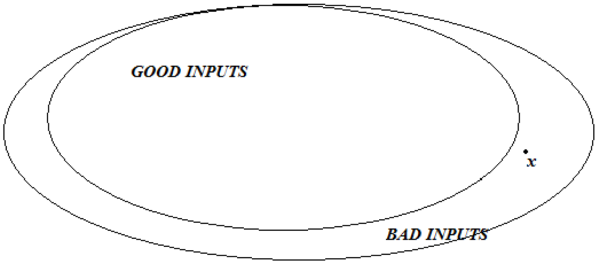


How to amplify undecidability?

Theorem (Miasnikov, Rybalov, 2008)

Let $C : I \to P(J)$ be a non-negligible cloning and $S \subseteq I$ is undecidable. Then $C(S)$ is super undecidable.

Idea of proof



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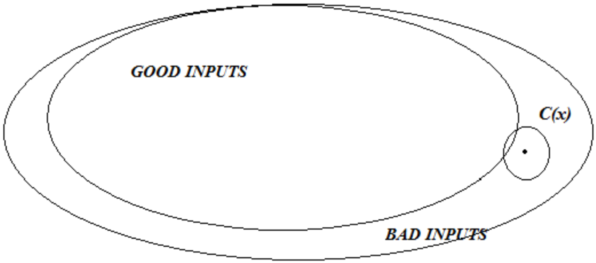

Example 1: the Halting Problem

Halting Problem

- Input: A Turing machine M over alphabet $\{0, 1\}$.
- Output:

$$\begin{array}{l} 1, \text{ if } M \text{ halts on } 0, \\ 0, \text{ if } M \text{ does not halt on } 0. \end{array}$$

Theorem (Turing)

The Halting Problem is algorithmically undecidable.

States

Non-final states $q_1, \ldots, q_n$. Final state q_0. Size of Turing machine is the number of non-final states.

Program

$$
\begin{aligned}
&(q_1, 0) \rightarrow (q_{j_1}, t_1, D_1), \\
&(q_1, 1) \rightarrow (q_{j_2}, t_2, D_2), \\
&\ldots \\
&(q_n, 1) \rightarrow (q_{j_{2n}}, t_{2n}, D_{2n})
\end{aligned}
$$

$t_i \in \{0, 1\}$, $D_i \in \{R, L\}$ – shift.

Normalized Turing machines

Definition

Turing machine M is **normalized**, if for every state q_i corresponding commands have the form

$$(q_i, 0) \to (q_j, t_1, D_1),$$
$$(q_i, 1) \to (q_k, t_2, D_2),$$

where $j, k \leq 2i + 1$.

Theorem

For every Turing machine there exists normalized Turing machine, computing the same function.

Generic Halting Problem

Theorem (Hamkins, Miasnikov)

The Halting Problem for Turing machines with one-way tape is generically decidable.

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Theorem (Rybalov)

The Halting Problem for normalized Turing machines is super undecidable.

Cloning

Cloning C adds new states $q_{n+1}, \ldots, q_{n+m}$ to machine M with program

$$\begin{aligned}
&(q_1, 0) \to (q_{j_1}, t_1, D_1),\\
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&\ldots\\
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\end{aligned}$$

and new arbitrary normalized commands for new states:

$$\begin{aligned}
&(q_{n+1}, 0) \to (q_{k_1}, u_1, E_1),\\
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Theorem

For every machine M the clone $C(M)$ is not negligible.

Example 2: First order theory of Arithmetic

Theory $Th(\mathbb{N})$

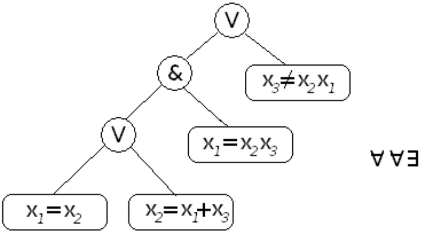
- Input: A first-order formula Φ of the language $\{+, -, \times, 0, 1\}$.
- Output:

$$\begin{cases} 1, & \text{if } \Phi \text{ is true over } \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Church)

$Th(\mathbb{N})$ is undecidable.

Representation of formulas



$$\forall x_1 \forall x_2 \exists x_3 (((x_1 = x_2) \vee (x_2 = x_1 + x_3))\&(x_1 = x_2 x_3)) \vee (x_3 \neq x_2 x_1)$$

Normalized representation

Definition

A formula representation is **normalized** if for every variable $x_i, i > 1$ there exists variable x_{i-1} which is the same leaf as x_i, or to the left leaf.

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Theorem

For every formula there exists equivalent normalized formula.

Generic first-order theory of Arithmetic

Theorem

$Th(\mathbb{N})$ is super undecidable.

Cloning of formulas

Cloning

For $\Phi = Q_1 x_1 \dots Q_t x_t \phi$ the clone $C(\Phi)$ is the set of formulas

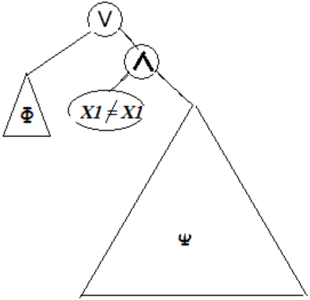
$$\Big\{ Q_1 x_1 \dots Q_t x_t Q_{t+1} x_{t+1} \dots Q_{3n} x_{3n} \Big(\phi \vee ((x_1 \neq x_1) \wedge \psi) \Big) \Big\},$$

where ψ is arbitrary quantifier-free normalized formula on $x_1, \dots, x_n$ and quantifiers $Q_{t+1}, \dots, Q_n$ are arbitrary.

Cloning of formulas

Theorem

For every formula Φ the set $C(\Phi)$ is not negligible.



Example 3: Word problem for semigroups

$S = \langle a_1, \dots, a_m | R \rangle$ – a semigroup with undecidable word problem.

Theorem (Myasnikov, Rybalov)

Word problem in the semigroup $S^+(x) = \langle A, x | R, xa_i = x, xx = x \rangle$ is super undecidable

Cloning of words

$$C(w_1, w_2) = (w_1 x v, w_2 x u),$$

where v and u are arbitrary words over alphabet $\{a_1, \dots, a_m, x\}$.